A CONNECTED STRING OF LONG THICK AND DOMINANTS

MARY REES

ABSTRACT. We prove that every Teichmuller geodesic of a finite type surface contains a string of intersecting *long, thick and dominant* segments, such that the distance between consecutive segments is bounded. This is key to obtaining some results about Teichmüller geodesics which mimic those for hyperbolic geodesics. These results have important applications to results about the geometry of hyperbolic three-manifolds.

1. Introduction

Teichmüller geodesics of finite type surfaces are very interesting objects. They are interesting in their own right, but their properties also have important applications. Two areas of applications come to mind. One area is dynamics, especially the dynamics of measured foliations, geodesic laminations and interval exchanges — and, of course the Teichmüller geodesic flow itself, and the dynamics of the mapping class group action on various boundaries — but the properties of Teichmüller geodesics do have dynamical implications at a much more basic level. The other application is to topology and geometry. Teichmüller space and Teichmüller geodesics were used to study the topology and geometry of spaces of critically finite branched coverings in [13]. Teichmüller geodesics have also been used to study the geometry of hyperbolic three-manifolds, especially those with finitely generated fundamental groups. The links between these two general areas of applications – to dynamics, and to the topology and geometry of particular spaces — are very strong.

In some respects, properties of Teichmüller geodesics mimic properties of hyperbolic geodesics. Indeed, the Teichmüller space for the torus with at most one puncture, or for a sphere with four punctures, is the hyperbolic plane, and the geodesics are the usual hyperbolic geodesics. For all higher type surfaces, the Margulis decomposition of a hyperbolic surface into "thick" and "thin" parts induces an approximate product metric structure on the corresponding "thin" parts of Teichmüller space, which conflicts with properties related to hyperbolic space. There are various strategies for dealing with this. One is to consider another space altogether, which has

stronger hyperbolic properties, such as the curve complex, which is Gromov hyperbolic [4, 5, 2]. Another strategy is to work directly with Teichmüller space, and to project to suitable coordinates to get hyperbolic properties. It is this strategy which is followed with the theory of "long thick and dominant" pieces. This theory was originally developed in [13]. The development there was for punctured spheres — simply because of the application for which it was intended. In fact, the theory works equally well for any finite type surface . A related theory, was developed by Rafi, in his thesis [9], with application to hyperbolic three-manifolds in mind. This theory has since been used extensively for example [10, 11, 12].

The purpose of this article is to present a result (3.7) about the long thick and dominant pieces on Teichmüller geodesics. Visually, the result is as follows. A Teichmüller geodesic is a path through hyperbolic surfaces of a fixed finite type. Each surface on the path has a Margulis decomposition into thick and thin subsurfaces. On each surface, there is either at least one "thick" piece, which is "dominant" – which can loosely be taken to mean that the Teichmüller distance is moving on this surface at about the same rate as the distance on the whole surface — or there is an annulus of large modulus on which the metric, induced by the quadratic differential associated to the quadratic differential, is approximately Euclidean – in which case the metric distance on this annulus is, once again, moving at approximately the rate of the distance on the whole surface. In each case, we call this subsurface "thick" —although an annulus of large modulus is in the thin part of the surface. For suitable parameters, these thick subsurfaces persist for some time along the geodesic. If they persist for a time which is regarded as sufficiently long, we call them *long, thick and dominant pieces.* One of the basic results of [13] was that long, thick and dominant pieces do exist, in any sufficiently long Teichmüller geodesic. If one were to make a three-dimensional model of the geodesic, out of wood, say, then these long thick and dominant pieces would look like chunky beads strung out along a necklace, with the cross-section of each bead in the shape of some subsurface. These beads could be very large. If a geodesic lies entirely in the thick part of Teichmüller space, then the model has a single bead, with cross-section in the shape of the whole surface, and running along the entire length of the necklace. But such geodesics are highly untypical. Precise quantification of the numbers and thicknesses of beads that one expects is not easy, but one certainly expects the number of beads to grow with the length of the geodesic, and that some of the beads will get more chunky with increasing length. In a broad sense, there is a huge and important literature related to this topic. The current purpose is concerned, more basically, with a property which holds for all geodesics. We will show that, for any

geodesic, or rather, for the necklace representation of any geodesic, there is a collection of beads, which cannot be moved more than a fixed length along the string without clashing against another bead in the collection. This means that the cross-sections of the adjacent beads in the collection intersect. It is then a consequence of the long, thick and dominant properties that the cross-sections of any two beads in the collection intersect.

It is natural to expect that this property of the necklace implies a certain rigidity about manifestations of paths in Teichmüller space which might not be geodesic in the strict sense, but have some related properties: quasi-geodesic, perhaps. Such manifestations occur in any hyperbolic manifold of dimension three with finitely generated fundamental group, travelling out from the core to any one of the ends. The beads in this case can be smoothed out of pleated surfaces. Topologically, the same structure appears in a Teichmuller geodesic. Because of the immovability of the beads, more than a certain distance, the geometric structure on the two collections of beads, up to bounded distortion, is the same: one in the three-manifold, one in the Teichmüller geodesic of surfaces. So a model of part of the three-manifold is obtained, in terms of the geometry of the Teichmüller geodesic. By an inductive procedure, the model can be extended to the whole manifold. Ultimately, the geometric structure of the hyperbolic three-manifold can be completely described, up to bounded distortion. A proof of the Ending Laminations Theorem follows. This proof can be found in [14].

The proof of 3.7 given in [14] does have some errors, but the spirit of the proof is unchanged. The proof is surprisingly difficult – or, at least, it has elements which have not occurred in other related results, like the simple existence of long thick and dominants. Roughly speaking, rather than looking for a set of positive measure on a surface, we look for a small interval on a surface — reducing to a point in the limit — arising as an intersection of a decreasing sequence of intervals. By the same method we could produce a Cantor set of zero measure rather than a point, but not a set of positive measure. It is not clear why the proof is so difficult, but the reason could be significant, because the corresponding result for curve complexes — the existence of a tight geodesic — is comparatively trivial.

2. Teichmuller space.

2.1. **Very basic objects in surfaces.** Unless otherwise stated, in this work, S always denotes an oriented finite type surface without boundary, that is, obtained from a compact oriented surface by removing finitely many points, called *punctures*. One does not of course need an explicit realisation of S as a compact minus finitely many points. One can simply take S to be a finite type surface. Up to homeomorphism, S is a compact minus

finitely many points, with each end of S mapped homeomorphically to a neighbourhood of the omitted point on the compact surface. A *multicurve* Γ on S is a union of simple closed nontrivial nonperipheral loops on S, which are isotopically distinct, and disjoint. A multicurve is *maximal* if it is not properly contained in any other multicurve. Of course, this simply means that the number of loops in the multicurve is $3g-3+b$, where g is the genus of S and b the number of punctures. A *gap* is a connected open subsurface α of a given surface S such that the topological boundary $\partial\alpha$ of α in S is a multicurve. If Γ is a multicurve on S, a *gap of* Γ is simply a component of $S \setminus (\cup\Gamma)$. If α is any gap, Γ is a *multicurve in* α if it satisfies all the above conditions for a closed surface, and, in addition, $\cup\Gamma \subset \alpha$ and no loops in Γ are homotopic to components of $\partial\alpha$. A positively oriented Dehn twist round a loop γ on an oriented surface S will always be denoted by τ_γ.

Let $\alpha_i \subset S$ be a gap or loop for $i = 1$, 2, isotoped so that $\partial\alpha_1$ and $\partial\alpha_2$ have only essential intersections, or with $\alpha_1 \subset \alpha_2$ if α_1 is a loop which can be isotoped into α_2. Then the *convex hull* $C(\alpha_1, \alpha_2)$ of α_1 and α_2 is the union of $\alpha_1 \cup \alpha_2$ and any components of $S \setminus (\alpha_1 \cup \alpha_2)$ which are topological discs with at most one puncture. Then $C(\alpha_1, \alpha_2)$ is again a gap or a loop. The latter only occurs if $\alpha_1 = \alpha_2$ is a loop. We are only interested in the convex hull up to isotopy, and it only depends on α_1 and α_2 up to isotopy. It is so called because, if α_i is chosen to have geodesic boundary, and $\tilde{\alpha}_i$ denotes the preimage of α_i in the hyperbolic plane covering S, then up to isotopy $C(\alpha_1, \alpha_2)$ is the projection to S of the convex hull of any component of $\tilde{\alpha_1} \cup \tilde{\alpha_2}$.

2.2. **Teichmüller space.** We consider Teichmüller space $\mathcal{T}(S)$ of a surface S. If $\varphi_i : S \to S_i = \varphi_i(S)$ is an orientation-preserving homeomorphism, and S_i is a complete hyperbolic surface with constant curvature -1, then we define the equivalence relation $\varphi_1 \sim \varphi_2$ if and only if there is an orientation-preserving isometry $\sigma : S_1 \to S_2$ such that $\sigma \circ \varphi_1$ is isotopic to φ_2. We define $[\varphi]$ to be the equivalence class of φ, and $\mathcal{T}(S)$ to be the set of all such $[\varphi]$, this being regarded as sufficient, since definition of a function includes definition of its domain. We shall often fix a complete hyperbolic metric of constant curvature -1 on S itself, which we shall also refer to as "the" Poincaré metric on S.

Complete hyperbolic structure in dimension two is equivalent to complex structure, for any orientable surface S of finite topological type and negative Euler characteristic, by the Riemann mapping theorem. So endowing such a surface S with a complex structure defines an element of the Teichmüller space $\mathcal{T}(S)$. More generally, the Measurable Riemann Mapping Theorem implies that supplying a bounded measurable conformal structure for S is

enough to define an element of $\mathcal{T}(S)$, and indeed $\mathcal{T}(S)$ is often (perhaps usually) defined in this way.

2.3. **Teichmüller distance.** We shall use d to denote Teichmüller distance, so long as the Teichmüller space $\mathcal{T}(S)$ under consideration is regarded as clear. Moreover a metric d will always be Teichmüller metric unless otherwise specified. If more than one space is under consideration, we shall use d_S to denote Teichmüller distance on $\mathcal{T}(S)$. The distance is defined as

$$d([\varphi_1],[\varphi_2]) = \inf\{\frac{1}{2}\log\|\chi\|_{qc} : [\chi\circ\varphi_1] = [\varphi_2]\},$$

where

$$\|\chi\|_{qc} = \|K(\chi)\|_\infty|, \quad K(\chi)(z) = \lambda(z)/\mu(z),$$

where $\lambda(z)^2 \geq \mu(z)^2 > 0$ are the eigenvalues of $D\chi_z^T D\chi_z$, and $D\chi_z$ is the derivative of χ at z (considered as a 2×2 matrix) and $D\chi_z^T$ is its transpose. The infimum is achieved uniquely at a map χ which is given locally in terms of a unique quadratic mass 1 differential $q(z)dz^2$ on $\varphi_1(S)$, and its *stretch* $p(z)dz^2$ on $\varphi_2(S)$. The local coordinates are

$$\zeta = x+iy = \int_{z_0}^{z}\sqrt{q(t)}dt,$$

$$\zeta' = \int_{z_0'}^{z'}\sqrt{p(t)}dt.$$

With respect to these local coordinates,

$$\chi(\zeta) = \chi(x+iy) = \lambda x + i\frac{y}{\lambda} = \zeta'.$$

So the distortion $K(\chi)(x+iy) = \lambda$ is constant. The singular foliations $x =$ constant and $y =$ constant on $\varphi(S)$ given locally by the coordinate $x+iy$ for $q(z)dz^2$ are known as the *stable and unstable foliations for* $q(z)dz^2$. We also say that $q(z)dz^2$ is the *quadratic differential at* $[\varphi_1]$ *for* $d([\varphi_1],[\varphi_2])$, and $p(z)dz^2$ is its *stretch* at $[\varphi_2]$.

2.4. **Thick and thin parts.** Let ε be any fixed Margulis constant for dimension two, that is, for any hyperbolic surface S, if $S_{<\varepsilon}$ is the set of points of S through which there is a nontrivial closed loop of length $<\varepsilon$, then $S_{<\varepsilon}$ is a (possibly empty) union of cylinders with disjoint closures. Then $(\mathcal{T}(S))_{<\varepsilon}$ is the set of $[\varphi]$ for which $(\varphi(S))_{<\varepsilon}$ contains an least one nonperipheral cylnder. The complement of $(\mathcal{T}(S))_{<\varepsilon}$ is $(\mathcal{T}(S))_{\geq\varepsilon}$. We shall sometimes write simply $\mathcal{T}_{<\varepsilon}$ and $\mathcal{T}_{\geq\varepsilon}$ if it is clear from the context which surface is meant. We shall also write $\mathcal{T}(\gamma,\varepsilon)$ for the set of $[\varphi]$ such that $(\varphi(S))_{<\varepsilon}$ contains a loop homotopic to $\varphi(\gamma)$. If Γ is a set of loops, we write

$$\mathcal{T}(\Gamma,\varepsilon) = \cup\{\mathcal{T}(\gamma,\varepsilon) : \gamma\in\Gamma\}.$$

2.5. **Length and the interpretation of Teichmüller distance.** We fix a surface S. It will sometimes be convenient to fix a hyperbolic metric on S, in which case we shall use $|\gamma|$ to denote length of a geodesic path γ with respect to this metric. With abuse of notation, for $[\varphi] \in \mathcal{T}(S)$ and a nontrivial nonperipheral closed loop γ on S, we write $|\varphi(\gamma)|$ for the length, with respect to the Poincaré metric on the hyperbolic surface $\varphi(S)$, of the geodesic homotopic to $\varphi(\gamma)$. We write $|\varphi(\gamma)|'$ for a modification of this, obtained as follows. We change the metric in ε_0-Margulis tube of $\varphi(S)$, for some fixed Margulis constant ε_0, to the Euclidean metric for this complex structure in the $\varepsilon_0/2$-Margulis tube, so that the loop round the annulus is length $\sqrt{|\varphi(\gamma)|}$, and a convex-linear combination with the Poincaré metric between the ε_0-Margulis tubes and $\varepsilon_0/2$-Margulis tubes. Then we take $|\varphi(\gamma)|'$ to be the length of the geodesic isotopic to $\varphi(\gamma)$ with respect to this modified metric. If the geodesic homotopic to $\varphi(\gamma)$ does not intersect any Margulis tube, then, of course, $|\varphi(\gamma)| = |\varphi(\gamma)|'$. Then for a constant C depending only on S and ε_0.

$$|\mathrm{Max}\{|\log|\varphi_2(\gamma)|' - \log|\varphi_1(\gamma)|'| : \gamma \in \Gamma\} - d([\varphi_1],[\varphi_2])| \leq C. \tag{2.5.1}$$

Here, Γ can be taken to be the set of all simple closed nonperipheral nontrivial closed loops on S. This estimate on Teichmüller distance derives from the fact that $|\varphi(\gamma)|'$ is inversely proportional to the largest possible square root of modulus of an embedded annulus in S homotopic to $\varphi(\gamma)$. See also 14.3, 14.4 and 14.7 of [13] (although the square root of modulus was mistakenly left out of [13]) but this estimate appears in other places, for example [5]. We can simply take Γ to be any set of simple closed nontrivial nonperipheral loops on S such that that every component of $S \setminus (\cup\Gamma)$ is a topological disc with at most one puncture. We shall call such a loop set *cell-cutting*

2.6. **Projections to subsurface Teichmüller spaces.** For any gap $\alpha \subset S$, we define a topological surface $S(\alpha)$ without boundary and a continuous map $\pi_\alpha : \mathcal{T}(S) \to \mathcal{T}(S(\alpha))$. We define $\varphi_\alpha(S(\alpha))$ by defining its conformal structure. After isotopy of φ, we can assume that all the components of $\varphi(\partial\alpha)$ are geodesic. We now write $\overline{\varphi(\alpha)}$ for the compactification of $\varphi(\alpha)$ obtained by cutting along $\varphi(\partial\alpha)$ and adding boundary components, each one isometric to some component of $\varphi(\partial\alpha)$. Then we form the Riemann surface $\varphi_\alpha(S(\alpha))$ by attaching a once-punctured disc $\{z : 0 < |z| \leq 1\}$ to $\overline{\varphi(\alpha)}$ along each component of $\varphi(\partial\alpha)$, taking the attaching map to have constant derivative with respect to length on the geodesics $\varphi(\partial\alpha)$ and length on the unit circle. Then we define $\varphi_\alpha = \varphi$ on α and then extend the map homeomorphically across each of the punctured discs. Then $[\varphi_\alpha]$ is a well-defined element of $\mathcal{T}(S(\alpha))$.

Now let α be a nontrivial nonperipheral simple closed loop. Fix an orientation on α Then we define

$$S(\alpha) = \overline{\mathbb{C}} \setminus \{\pm 2, \pm\frac{1}{2}\}.$$

Now we define an element $[\varphi_\alpha] = \pi_\alpha([\varphi]) \in \mathcal{T}(S(\alpha))$, for each $[\varphi] \in \mathcal{T}(S)$, as follows. Fix a Margulis constant ε. If $|\varphi(\alpha)| \leq \varepsilon$, let A be the closed ε-Margulis tube in $\varphi(S)$ homotopic to $\varphi(\alpha)$. If $|\varphi(\alpha)| > \varepsilon$, let A be the closed η-neighbourhood of the geodesic homotopic to $\varphi(\alpha)$ where η is chosen so that A is an embedded annulus, and thus can be chosen bounded from 0 if $|\varphi(\alpha)|$ is bounded above. Fix a simple closed geodesic $\beta(\alpha)$ which intersects α at most twice and at least once, depending on whether or not α separates S. We can assume after isotopy that $\varphi(\alpha)$ and $\varphi(\beta(\alpha))$ are both geodesic, and we fix a point $x_1(\alpha) \in \varphi(\alpha \cap \beta(\alpha))$. We make a Riemann surface S_1 homeomorphic to the sphere, by attaching a unit disc to each component of ∂A, taking the attaching maps to have constant derivative with respect to length. Then we define φ_α to map $\overline{\mathbb{C}}$ to S_1 by mapping $\{z : |z| = 1\}$ to $\varphi(\alpha)$, 1 to $x_1(\alpha)$, $\{z : \frac{1}{2} \leq |z| \leq 2\}$ to A and $\{z \in \mathbb{R} : \frac{1}{2} \leq z \leq 2\}$ to the component of $\varphi(\beta(\alpha)) \cap A$ containing α. Then $\varphi_\alpha(S(\alpha))$ is a four-times punctured sphere and so we have an element $[\varphi_\alpha] \in \mathcal{T}(S(\alpha))$. Now the Teichmüller space $\mathcal{T}(S(\alpha))$ is isometric to the upper half plane H^2 with metric $\frac{1}{2}d_P$, where d_P denotes the Poincaré metric $\dfrac{dx^2 + dy^2}{y^2}$. This is well-known. We now give an identification. Let $n_\alpha([\varphi]) = n_{\alpha,\beta(\alpha)}([\varphi])$ be the integer assigning the minimum value to

$$m \to |\varphi(\tau_\alpha^m(\beta(\alpha))|.$$

If there is more than one such integer then we take the smallest one. There is a bound on the number of such integers of at most two consecutive ones. We see this as follows. Let ℓ be a geodesic in the hyperbolic plane and let g be a Möbius involution such that $g.\ell$ is disjoint from, and not asymptotic, to ℓ, and such that the common perpendicular geodesic segment from ℓ to $g.\ell$ meets them in points x_0, $g.x_0$, for some $x_0 \in \ell$. Then the complete geodesics meeting both ℓ and $g.\ell$ and crossing them both at the same angle, are precisely those that pass through points x and $g.x$ for some $x \in \ell$, and the hyperbolic length of the segment joining x and $g.x$ increases strictly with the length between x_0 and x. This implies the essential uniqueness of n, as follows. We take ℓ to be a lift of $\varphi(\alpha)$ to the universal cover, and let ℓ_1 be another lift of $\varphi(\alpha)$, such that some lift of $\varphi(\beta(\alpha))$ has endpoints on ℓ and ℓ_1. Then g is determined by making $\ell_1 = g.\ell$ for g as above. But also $\ell_1 = g_2.\ell$, where g_2 is the element of the covering group corresponding to $\varphi(\beta(\alpha))$. We also have an element g_1 of the covering group corresponding

to $\varphi(\alpha)$, which preserves ℓ and orientation on ℓ. Then $|\varphi(\tau_\alpha^m(\beta(\alpha)))|$ is the distance between x and $g.x$ for the unique x such that some lift of a loop freely homotopic to $\varphi(\tau_\alpha^m(\beta(\alpha)))$ has endpoints at x and $g.x$. The endpoints are $g_1^{-m}.y$ and $g_2.y$ for y such that $x = g_1^{-m}.y$. So x is determined by the $y = y_m$ such that $g.x = g_2.g_1^m.x$. So then $d(x, x_0) = \frac{1}{2}d(x_0, g^{-1}g_2g_1^m.x_0)$, which takes its minimum at either one, or two adjacent, values of m.

Then the isometric identification with H^2 can be chosen so that, if we use the identification to regard π_α as a map to H^2,

$$\pi_\alpha([\varphi]) = n_\alpha([\varphi]) + i|\varphi(\alpha)|^{-1} + O(1). \tag{2.6.1}$$

If α is either a gap or a loop we now define a semimetric d_α by

$$d_\alpha([\varphi_1], [\varphi_2]) = d_{S(\alpha)}(\pi_\alpha([\varphi_1]), \pi_\alpha([\varphi_2])).$$

3. Teichmüller geodesics: long thick and dominant definitions.

In this section we explain and expand some of the ideas of long thick and dominant (ltd) segments of geodesics in Teichmüller space $\mathcal{T}(S)$ which were used in [13]. The theory of [13] was explicitly for marked spheres only, because of the application in mind, but in fact the theory works, without adjustment, for any finite type surface, given that projections π_α to smaller Teichmüller spaces $\mathcal{T}(S(\alpha))$ for subsurfaces α of S have been defined in 2.6. For proofs, for the most part, we refer to [13]. The basic idea is to get into a position to apply arguments which work along geodesics which never enter the thin part of Teichmüller space, by projecting to suitable subsurfaces α using the projections π_α of 2.6. We use the basic notation and theory of Teichmüller space $\mathcal{T}(S)$ from Section 2.

3.1. **Good position.** Let $[\varphi] \in \mathcal{T}(S)$. Let $q(z)dz^2$ be a quadratic differential on $\varphi(S)$. All quadratic differentials, as in 2.3, will be of total mass 1. Let γ be a nontrivial nonperipheral simple closed loop on S. Then there is a limit of isotopies of $\varphi(\gamma)$ to *good position* with respect to $q(z)dz^2$, with the limit possibly passing through some punctures. If γ is the isotopy limit, then either γ is at constant angle to the stable and unstable foliations of $q(z)dz^2$, or is a union of segments between singularities of $q(z)dz^2$ which are at constant angle to the stable and unstable foliations, with angle $\geq \pi$ between any two consecutive segments at a singularity, unless it is a puncture. An equivalent statement is that γ is a geodesic with respect to the singular Euclidean metric $|q(z)|d|z|^2$. If two good positions do not coincide, then they bound an open annulus in $\varphi(S)$ which contains no singularities of $q(z)dz^2$. See also 14.5 of [13].

The *q-d length* $|\varphi(\gamma)|_q$ is length with respect to the quadratic differential metric for any homotopy representative in good position. (See 14.5 of [13].)

We continue, as in Section 2, to use $|\varphi(\gamma)|$ to denote the hyperbolic, or Poincaré, length on $\varphi(S)$ of the geodesic on $\varphi(S)$ homotopic to $\varphi(\gamma)$. If $[\varphi] \in \mathcal{T}_{\geq\varepsilon}$ then there is a constant $C(\varepsilon) > 0$ such that for all nontrivial nonperipheral closed loops γ,

$$\frac{1}{C(\varepsilon)} \leq \frac{|\varphi(\gamma)|_q}{|\varphi(\gamma)|} \leq C(\varepsilon).$$

We also define $|\varphi(\gamma)|_{q,+}$ to be the integral of the norm of the projection of the derivative of $\varphi(\gamma)$ to the tangent space of the unstable foliation of $q(z)dz^2$, and similarly for $|\varphi(\gamma)|_{q,-}$. So these are both majorised by $|\varphi(\gamma)|_q$, which is, in turn, majorised by their sum.

3.2. **Area.** The following definitions come from 9.4 of [13]. For any essential nonannulus subsurface $\alpha \subset S$, $a(\alpha, q)$ is the area with respect to $q(z)dz^2$ of $\varphi(\alpha)$ where $\varphi(\partial\alpha)$ is in good position and bounds the smallest area possible subject to this restriction. If α is a loop at x then $a(\alpha, q)$ is the smallest possible area of an annulus of modulus 1 and homotopic to $\varphi(\alpha)$. We are only interested in this quantity up to a bounded multiplicative constant. It is boundedly proportional to $|\varphi(\alpha)|_q^2$ whenever $\varphi(\alpha)$ is in good position, and $|\varphi(\alpha)|$ is bounded. We sometimes write $a(\alpha, x)$ or even $a(\alpha)$ for $a(\alpha, q)$, if it is clear from the context what is meant.

Generalising from 3.1, there is a constant $C(\varepsilon)$ such that, if $\varphi(\alpha)$ is homotopic to a component of $(\varphi(S))_{\geq\varepsilon}$, then for all nontrivial nonperipheral non-boundary-homotopic closed loops $\gamma \in \alpha$,

$$\frac{1}{C(\varepsilon)}|\varphi(\gamma)| \leq \frac{|\varphi(\gamma)|_q}{\sqrt{a(\alpha, q) + a(\partial\alpha, q)}} \leq C(\varepsilon)|\varphi(\gamma)|.$$

Now suppose that ℓ is a directed geodesic segment in $\mathcal{T}(S)$ containing $[\varphi]$, and that $q(z)dz^2$ is the quadratic differential at $[\varphi]$ for $d([\varphi], [\psi])$ for any $[\psi]$ in the positive direction along ℓ from $[\varphi]$ (see 2.2.) Let $p(z)dz^2$ be the stretch of $q(z)dz^2$ at $[\psi]$, and let χ be the minimum distortion map with $[\chi \circ \varphi] = [\psi]$. Then χ maps the q-area element to the p-area element. Then $a(\alpha, q) = a(\alpha, p)$ if α is a gap, but if α is a loop, $a(\alpha, y)$ varies for $y \in \ell$.

If α is a gap, we define $a'(\alpha) = a(\alpha)$. Now suppose α is a loop. We define $a'(\alpha, [\varphi], q)$ (or simply $a'(\alpha)$ if the context is clear) to be the q-area of the largest modulus annulus (possibly degenerate) homotopic to $\varphi(\alpha)$ and with boundary components in good position for $q(z)dz^2$. Then in both cases, gap and loop, $a'(\alpha)$ is constant along the geodesic determined by $q(z)dz^2$.

3.3. **The long thick and dominant definition.** Now we fix parameter functions Δ, r, $s : (0, \infty) \to (0, \infty)$ and a constant K_0.

Let α be a gap. Let ℓ be a geodesic segment. We say that α is *long, ν-thick and dominant at* x (for ℓ, and with respect to (Δ, r, s)) if $x \in \ell$ is the

centre of a segment ℓ_1 in the geodesic extending ℓ of length $2\Delta(\nu)$ such that $|\psi(\gamma)| \geq \nu$ for all $[\psi] \in \ell_1$ and nontrivial nonperipheral $\gamma \subset \alpha$ not homotopic to boundary components, but $\ell_1 \subset \mathcal{T}(\partial\alpha, r(\nu))$ and $a(\partial\alpha, y) \leq s(\nu)a(\alpha, y)$ for all $y \in \ell_1$. We shall then also say that α is long ν-thick and dominant along ℓ_1. See 15.3 of [13].

A loop α at x is K_0-*flat at* $x = [\varphi]$ (for ℓ) if $a'(\alpha) \geq K_0 a(\alpha)$. This was not quite the definition made in [13] where the context was restricted to S being a punctured sphere, but the results actually worked for any finite type surface. The term arises because if α is K_0-flat then the metric $|q(z)|dz^2$ is equivalent to a Euclidean (flat) metric on an annulus homotopic to $\varphi(\alpha)$ of modulus $K_0 - O(1)$. For fixed K_0 we may simply say *flat* rather than K_0-flat.

In future, we shall often refer to parameter functions as quadruples of the form (Δ, r, s, K_0). For a fixed quadruple, we shall refer to (α, ℓ) as *ltd* if either α is a gap which is long, ν- thick and dominant along ℓ with respect to these parameter functions, or α is a loop which is K_0-flat along ℓ.

If (α, ℓ) is ltd, and α is a gap, then $d_\alpha(x, y)$ is very close to $d(x, y)$ for all x, $y \in \ell$. This is a consequence of the results of Section 11 of [13]. For us, here, the fact that the two quantities differ by some additive constant is sufficient motivation. It is also probably worth noting (again by the results of Chapter 11 of [13]) that if $[\varphi] \in \ell$ and $\pi_\alpha([\varphi]) = [\varphi_\alpha]$, then $\varphi_\alpha(S(\alpha))$ and the component $S(\alpha, r(\nu), [\varphi])$ of $(\varphi(S))_{\geq r(\nu)}$ homotopic to $\varphi(\alpha)$ are isometrically very close, except in small neighbourhoods of some punctures, and the quadratic differentials $q(z)dz^2/a(\alpha)$ at $[\varphi]$ for $d([\varphi], [\psi])$ ($[\psi] \in \ell$) and the quadratic differential $q_\alpha(z)dz^2$ at $[\varphi_\alpha]$ for $d_\alpha([\varphi], [\psi])$, are very close. We refer to [13] for the main applications of ltd's, including a thin triangle result.

3.4. We now show that ltd's exist, in some abundance. This was the content of the first basic result about ltd's in 15.4 of [13], which was stated only for S being a punctured sphere, but the proof worked for a general finite type surface.

Lemma *For some ν_0 and Δ_0 depending only on (Δ, r, s, K_0) (and the topological type of S), the following holds. Any geodesic segment ℓ of length $\geq \Delta_0$ contains a segment ℓ' for which there is α such that one of the following holds.*

- *α is a gap which is long ν-thick and dominant along ℓ' for some $\nu \geq \nu_0$ and $a(\alpha) \geq 1/(-2\chi(S) + 1) = c(S)$ (where χ denotes Euler characteristic.*
- *α is a K_0-flat loop along ℓ' and $a(\alpha) \geq c_0$, where c_0 depends only on the topological type of S and the ltd parameter functions*

More generally, there is s_0 depending only on (Δ, r, s, K_0) (and the topological type of S) such that, whenever $\omega \subset S$ is such that $a(\partial\omega) \leq s_0 a(\omega)$, then we can find α as above with $\alpha \subset \omega$ and $a(\alpha) \geq c_0 a(\omega)$.

Proof. (See also 15.4 of [13].) We consider the case $\omega = S$. Write $r_1(\nu) = e^{-\Delta(\nu)} r(\nu)$. Let $g = -2\chi(S) \geq 2$ and let r_1^g denote the g-fold iterate. We then take

$$\nu_0 = r_1^g(\varepsilon_0)$$

for a fixed Margulis constant ε_0 and we define

$$\Delta_0 = 2\sum_{j=1}^{g} \Delta(r_1^j(\varepsilon_0)).$$

Then for some $j \leq g$, we can find $\nu = r_1^j(\varepsilon_0)$ and $[\varphi] \in \ell$ such that the segment ℓ' of length $2\Delta(\nu)$ centred on $y = [\varphi]$ is contained in ℓ, and such that for any nontrivial nonperipheral loop γ, either $|\varphi'(\gamma)| \geq \nu$ for all $[\varphi'] \in \ell'$, or $|\varphi(\gamma)| \leq r_1(\nu)$ — in which case $|\varphi'(\gamma)| \leq r(\nu)$ for all $[\varphi'] \in \ell'$. For any loop γ with $|\varphi(\gamma)| < \varepsilon_0$, if β is a gap such that $\gamma \subset \partial\beta$ and there is a component of $(\varphi(S))_{\geq \varepsilon_0}$ homotopic to $\varphi(\beta)$ and separated from the flat annulus homotopic to $\varphi(\gamma)$ by an annulus of modulus Δ_1, we have, since every zero of $q(z)dz^2$ has order at most $2g$, for a constant C_1 depending only on the topological type of S,

$$C_1^{-1} a(\gamma, [\varphi]) e^{\Delta_1} \leq a(\beta) \leq C_1 a(\gamma, [\varphi]) e^{(2g+2)\Delta_1}. \tag{3.4.1}$$

Now let β be a subsurface such that $\varphi(\beta)$ is homotopic to a component $S(\beta, \nu)$ of $(\varphi(S))_{\geq \nu}$ or $(\varphi(S))_{\geq \nu}$ such that $a'(\beta) + a'(\partial\beta)$ is of maximal area, where $\beta = \partial\beta$ if β is a loop. This means that

$$a'(\beta) + a'(\partial\beta) \geq \frac{1}{3g},$$

with $1/3g$ replaced by $1/g$, if β is a gap. Then by (3.4.1), we have a bound of $O(e^{(2g^2+2g)/\nu})$ on the ratio of areas of any two components of $(\varphi(S))_{\geq \varepsilon_0}$ in $S(\beta, \nu)$. If there is a component γ of $\partial\beta$ such that

$$a'(\gamma) \geq e^{-1/(9g r_1(\nu))} a'(\beta),$$

then we take $\gamma = \alpha$, and γ is K_0-flat at $[\varphi]$, assuming that $r(\nu)$ is sufficiently small, for all ν given K_0, and the lower bound on $a'(\beta)$ gives the required lower bound of $a'(\gamma) \geq c_0$, for c_0 depending only on ν_0, that is, only on (Δ, r) and the topological type of S. Otherwise, we take $\beta = \alpha$, and we have, for all $y' \in \ell'$,

$$a(\partial\alpha, y') \leq e^{\Delta(\nu)} e^{-1/(9gr(\nu))} a(\alpha).$$

Assuming $r(\nu)$ is sufficiently small given $s(\nu)$ and $\Delta(\nu)$, α is long ν-thick and dominant along ℓ' for (Δ, r, s), and $a(\alpha) \geq 1/(g+1)$, as required.

The case $\omega = S$ is similar. We only need s_0 small enough for $a(\partial\omega)/a(\omega)$ to remain small along a sufficiently long segment of ℓ. $\square$

Because of this result, we can simplify our notation. So let ν_0 be as above, given (Δ, r, s, K_0). We shall simply say α is *ltd* (at x, or along ℓ_1) if either α is a gap and long ν-thick and dominant for some $\nu \geq \nu_0$, or α is a loop and K_0-flat, at x or along ℓ_1. We shall also say that (α, ℓ_1) is ltd, and, if we want to be more specific, we shall say that (α, ℓ_1) is ltd with respect to $(\Delta, r, s, K_0, \nu_0)$.

3.5. We refer to Chapters 14 and 15 of [13] for a summary of all the results concerning ltd's, where, as already stated, the context is restricted to S being a punctured sphere, but the results work for any finite type surface. The main points about ltd's are, firstly, that they are good coordinates, in which arguments which work in the thick part of Teichmüller space can be applied, and secondly that there is only bounded movement in the complement of ltd's. This second fact is worth scrutiny. It is, at first sight, surprising. It is proved in 15.14 of [13], which we now state, actually slightly corrected since short interior loops in α were forgotten in the statement there (although the proof given there does consider short interior loops) and slightly expanded in the case of α being a loop.

Lemma *Fix long thick and dominant parameter functions (Δ, r, s, K_0), and let $\nu_0 > 0$ also be given and sufficiently small. Then there exists $L = L(\Delta, r, s, K_0, \nu_0)$ such that the following holds. Let ℓ be a geodesic segment and let $\ell_1 \subset \ell$ and let $\beta \subset S$ be a maximal subsurface up to homotopy with the property that $\beta \times \ell_1$ is disjoint from $\alpha \times \ell'$ for all ltd's (α, ℓ') with respect to $(\Delta, r, s, K_0, \nu_0)$. Suppose also that all components of $\partial\beta$ are nontrivial nonperipheral. Then β is a disjoint union of gaps and loops β_1 such that the following hold.*

$$|\varphi(\partial\beta_1)| \leq L \text{ for all } [\varphi] \in \ell_1. \tag{3.5.1}$$

If β_1 is a gap, then for all $[\varphi]$, $[\psi] \in \ell_1$ and nontrivial nonperipheral non-boundary-parallel closed loops γ in β_1,

$$L^{-1} \leq \frac{|\varphi(\gamma)|'}{|\psi(\gamma)|'} \leq L, \tag{3.5.2}$$

$$|\varphi(\gamma)| \geq L^{-1}. \tag{3.5.3}$$

If β_1 is a loop, then for all $[\varphi]$, $[\psi] \in \ell_1$,

$$|\mathrm{Re}(\pi_{\beta_1}([\varphi]) - \pi_{\beta_1}([\psi]))| \leq L. \tag{3.5.4}$$

Also if γ is in the interior of β, and $\ell_1 = [[\varphi_1],[\varphi_2]]$, then given $\varepsilon_1 > 0$ there exists $\varepsilon_2 > 0$ depending only on ε_1 and $(\Delta, r, s, K_0, \nu_0)$ such that

$$\begin{aligned}&\text{If } |\varphi(\gamma)| < \varepsilon_2, \text{ then}\\ &\mathrm{Min}(|\varphi_1(\gamma)|, |\varphi_2(\gamma)|) \le \varepsilon_1, \text{ and}\\ &\mathrm{Max}(|\varphi_1(\gamma)|, |\varphi_2(\gamma)|) \le L.\end{aligned} \tag{3.5.5}$$

If (3.5.1), and either (3.5.2) and (3.5.3), or (3.5.4) hold for (β_1, ℓ_1), depending on whether β_1 is a gap or a loop, we say that (β_1, ℓ_1) is *bounded* (by L). Note that L depends on the ltd parameter functions, and therefore is probably extremely large compared with $\Delta(\nu)$ for many values of ν, perhaps even compared with $\Delta(\nu_0)$.

Here are some notes on the proof. For fuller details, see 5.14 of [13]. First of all, under the assumption that $\partial\beta$ satisfies the condition (3.5.1), it is shown that β is a union of β_1 satisfying (3.5.1) to (3.5.3). First, we show that (3.5.2) holds for all $\gamma \subset \beta$ such that $|\varphi_i(\gamma)|$ is bounded from 0 for $i = 1$, 2, and that (3.5.5) holds for β. This is done by breaking ℓ into three segments, with $a'(\beta)$ dominated by $|\varphi(\partial\beta)|_q^2$ on the two outer segments ℓ_-, ℓ_+, where $q(z)dz^2$ is the quadratic differential at $[\varphi]$ for ℓ. The middle segment $[[\varphi_-],[\varphi_+]]$ has to be of bounded length by the last part of 3.4, since there are no ltd's in β along ℓ. Then $|\varphi(\partial\beta)|_q$ is boundedly proportional to $|\varphi(\partial\beta)|_{q,-}$ along ℓ_-, and to $|\varphi(\partial\beta)|_{q,+}$ along ℓ_+. We can obtain (3.5.2) along ℓ_+, at least for a nontrivial $\beta' \subset \beta$ for which we can "lock " loops $\varphi(\gamma)$, for which $|\varphi(\gamma)|$ is bounded, along stable segments to $\varphi(\partial\beta)$. If $\beta' \neq \beta$ and $\gamma' \subset \partial\beta'$ is in the interior of β, then either $|\varphi_+(\gamma')|$ is small, or $|\varphi_+(\gamma')|_{q+}$ is dominated by $|\varphi_+(\gamma')|_{q+,-}$, where $q_+(z)dz^2$ is the stretch of $q(z)dz^2$ at $[\varphi_+]$. In the case when $|\varphi_+(\gamma')|$ is small, there is some first point $[\varphi_{++}] \in \ell_+$ for which $|\varphi_{++}(\gamma')|_{q_{++}}$ is dominated by $|\varphi_{++}(\gamma')|_{q_{++},-}$, where $q_{++}(z)dz^2$ is the stretch of $q(z)dz^2$ at $[\varphi_{++}]$. For this point, $|\varphi_{++}(\gamma')|$ is still small, and can be locked to a small segment of $\varphi_{++}(\partial\beta)$. This means that we can deduce that $|\varphi_2(\gamma')|$ is small, giving (3.5.5). So one proceeds by induction on the topological type of β, obtaining (3.5.2) and (3.5.5) for β from that for $\beta \setminus \beta'$. Then (3.5.5) and (3.5.2) imply that the set of loops with $|\varphi_1(\gamma)| < \varepsilon_1$ or $|\varphi_2(\gamma)| < \varepsilon_1$, for a sufficiently small ε_1, do not intersect transversally. This allows for a decomposition into sets β_1 satisfying (3.5.1), (3.5.2) and (3.5.3). One then has to remove the hypothesis (3.5.1) for $\partial\beta$. This is done by another induction, considering successive gaps and loops β' disjoint from all ltd's along segments ℓ' of ℓ, with $|\varphi(\partial\beta')| \le \varepsilon_0$ for $[\varphi] \in \ell'$, possibly with $\partial\beta' = \emptyset$. One then combines the segments and reduces the corresponding β', either combining two at a time, or a whole succession together, if the β' are the same along a succession of segments. In finitely many steps, one reaches (β, ℓ) finding in the process that $\partial\beta$ does satisfy (3.5.1).

As for showing that β satisfies (3.5.4), that follows from the following lemma — which is proved in 15.13 of [13], but not formally stated. Note that if β is a loop, $a'(\beta,[\varphi])$ is constant for $[\varphi]$ in a geodesic segment ℓ, but $a(\beta,q)$ is proportional to $|\varphi(\beta)|_q^2$ (if $q(z)dz^2$ is the quadratic differential at $[\varphi]$ for ℓ), which has at most one minimum on the geodesic segment and otherwise increases or decreases exponetially with distance along the segment, depending on whether $|\varphi(\beta)|_q$ is boundedly proportional to $|\varphi(\beta)|_{q,+}$ or $|\varphi(\beta)|_{q,-}$. So for any K_0, the set of $[\varphi]\in\ell$ for which $a'(\beta)\geq K_0a(\beta,[\varphi])$ is a single segment, up to bounded distance. This motivates the following.

Lemma 3.6. *Given K_0, there is $C(K_0)$ such that the following holds. Let ℓ be any geodesic segment. Suppose that β is a loop and $a'(\beta)\leq K_0a(\beta,[\varphi])$ for all $[\varphi]\in\ell$. Then for all $[\varphi]$, $[\psi]\in\ell$,*

$$|\mathrm{Re}(\pi_\beta([\varphi])-\pi_\beta([\psi]))|\leq C(K_0).$$

Proof. The argument is basically given in 15.13 of [13]. Removing a segment of length bounded in terms of K_0, ε_0 at one end, we obtain a reduced segment ℓ' such that that $a'(\beta)\leq\varepsilon_0a(\beta,[\varphi])$ for all $[\varphi]\in\ell'$. We use the quantity $n_\beta([\varphi])$ of 2.6, which is $\mathrm{Re}(\pi_\beta([\varphi]))+O(1)$ and is given to within length $O(1)$ by m minimising $|\varphi(\tau_\beta^m(\zeta))|$ for a fixed ζ crossing β at most twice (or a bounded number of times). This is the same to within $O(1)$ as the m minimising $|\varphi(\tau_\beta^m(\zeta))|_q$ for any quadratic differential $q(z)dz^2$. (To see this, note that the shortest paths, in the Poincaré metric, across a Euclidean annulus $\{z:r<|z|<1\}$, are the restrictions of straight lines through the origin.) Assume without loss of generality that $|\varphi(\beta)|_q$ is boundedly proportional to $|\varphi(\beta)|_{q,+}$ for $[\varphi]\in\ell$, and $q(z)dz^2$ the quadratic differential at $[\varphi]$ for ℓ. The good positions of $\varphi(\tau_\beta^m(\zeta))$ for all m are locked together along stable segments whose qd-length is short in comparison with $|\varphi(\beta)|_q$, if ε_0 is sufficiently small. So $n_\beta([\varphi])$ varies by <1 on ℓ, and is thus constant on ℓ', if ε_0 is sufficiently small, and hence varies by at most $C(K_0)$ on ℓ.□

3.7. **A Chain of ltd's.** Now the result we are aiming for is the following. The proof of this result is different in character from that of 3.4, being, essentially, a construction of a zero measure Cantor set, while 3.4 obtained a set with a lower bound on area. This result is in any case more sophisticated, because it uses 3.5 — and hence also 3.4 — in the course of the proof. This result can be regarded as a parallel to the existence of a tight geodesic in the curve complex used by Minsky et al.. [6, 3]. For reasons which are not entirely clear to me, but which may be significant, this result appears to be much harder to prove.

Theorem *Fix long thick and dominant parameter functions and flat constant (Δ,r,s,K_0). Then there exist Δ_0, $\delta_0>0$ and ν_0 depending only on*

(Δ, r, s, K_0) and the topological type of S such that the following holds. Let $[y_0, y_T] = [[\varphi_0], [\varphi_T]]$ be any geodesic segment in $\mathcal{T}(S)$ of length $T \geq \Delta_0$, parametrised by length. Then there exists a sequence (α_i, ℓ_i) $(1 \leq i \leq R_0)$ such that:

- *(α_i, ℓ_i) is ltd with respect to (Δ, r, s, K_0), and ν-thick for some $\nu \geq \nu_0$ if α_i is a gap;*
- *$(\alpha_i, \ell_i) < (\alpha_{i+1}, \ell_{i+1})$ for $i < R_0$, where the ordering $<$ is as in 4.5, that is, ℓ_i ends before ℓ_{i+1} starts, and $\alpha_i \cap \alpha_{i+1} \neq \emptyset$;*
- *each segment of $[y_0, y_T]$ of length Δ_0 intersects some ℓ_i;*
- *$a'(\alpha_i) \geq \delta_0$.*

4. The proof of Theorem 3.7.

4.1. **Idea of the proof.** Throughout this chapter, we let $\{y_t : t \in \mathbb{R}\}$ be a fixed Teichmüller geodesic, containing the segment $[y_0, y_T]$ which is the subject of Theorem 3.7. Here, t parametrises length with respect to the Teichmüller metric. We write $y_t = [\varphi_t] = [\chi_t \circ \varphi_0]$, with χ_t minimising distortion, with quadratic differential $q_0(z)dz^2$ and y_0 and stretch $q_t(z)dz^2$ at y_t (see also 2.3). We shall write $|.|_t$ for $|.|_{q_t}$ and $|\cdot|_{t,+}$ and $|\cdot|_{t,-}$ for the unstable and stable lengths $|\cdot|_{q_t,+}$ and $|\cdot|_{q_t,-}$ respectively. (See 3.1 for definitions.)

We shall prove Theorem 3.7 by showing that, if Δ_0 is sufficiently large, for some $t_1 \in [0, \Delta_0]$ and for some segment of unstable segment $\zeta_1 \subset \varphi_0(\alpha_1)$, and some $\xi \in \zeta_1$, for every Teichmüller geodesic segment ℓ of length Δ_0, along $[y_0, y_T]$, there is an ltd (α_i, ℓ_i) with $\ell_i \subset \ell$ and $\xi \in \varphi_0(\alpha_i)$. It then follows that the intersection of all such $\varphi_0(\alpha_i)$ is nonempty, and therefore any two such α_i intersect essentially. It is not the case that any sequence (α_i, ℓ_i) for $i \leq j$ is extendable, and this is the main obstacle that we have to overcome.

All the ltd's (α_j, ℓ_j) have to be viewed in terms of $\zeta_1 \cap \varphi_0(\alpha_j)$. In particular, we need to choose α_{j+1} so that

$$\varphi_0(\alpha_j) \cap \varphi_0(\alpha_{j+1}) \cap \zeta_1 \neq \emptyset.$$

We transfer this to showing that, for a suitable t_j such that $|\varphi_{t_j}(\partial\alpha_j)|$ is bounded at $[\varphi_{t_j}]$, and for a given $\zeta_j \subset \zeta_1 \cap \varphi_0(\alpha_j)$,

$$\varphi_{t_j}(\alpha_j) \cap \varphi_{t_j}(\alpha_{j+1}) \cap \chi_{t_j}(\zeta_j) \neq \emptyset.$$

Basically, we will need to choose α_{j+1} intersecting α_j so that $\varphi_{t_j}(\alpha_{j+1}) \cap \chi_{t_j}(\zeta_j)$ is in the complement of a "bad subset" of $\varphi_{t_j}(\alpha_j) \cap \chi_{t_j}(\zeta_j)$, where the "bad subset" contains all $\varphi_{t_j}(\beta) \cap \chi_{t_j}(\zeta_j)$ such that β is bounded along a suitably chosen sufficiently long Teichmüller geodesic segment $\ell \subset [y_{t_j}, y_{t_j+\Delta_0}]$. This requires showing that the bad subset is sufficiently small in a useful

sense. One thing that we can do, as we shall see, is to bound $a'(\beta)$ for these β. We use $a'(\beta)$ rather than $a(\beta)$, as $a'(\beta)$ records the area of a subset of $\varphi_t(S)$ which is constant as t varies. But this is not enough, because an area bound is clearly not enough to bound length of intersection with an arc $\chi_{t_j}(\zeta_j) \subset \varphi_{t_j}(S)$. We need to bound the set of such $\varphi_{t_j}(\beta) \cap \chi_{t_j}(\zeta_j)$ within a set of a certain shape: a union of intervals which are sufficiently short, and bounded apart by intervals which are sufficiently long. It turns out that we can do this, provided we enlarge the bad set in a certain way. The method involves a careful comparison between Poincaré distance along the surfaces of the Teichmüller geodesic $[y_0, y_T]$, and the distance determined by the different quadratic differentials $q_t(z)dz^2$. In particular, we will use properties of the graph of the *qd-length function* $\log |\varphi_t(\partial\beta)|_t$. We will do this in 4.6.

But first we recall the partial order properties of ltd's, which are derived as follows. The whole of the theory of ltd gaps and loops is based on a simple dynamical lemma which quantifies density of leaves of the stable and unstable foliations of a quadratic differential. This is basically 15.11 of [13]. But the statement is slightly more general.

Lemma 4.2. *Let a deceasing function $\varepsilon : (0,\infty) \to (0,\infty)$ be given. Then there is a function $L : (0,\infty) \times (0,\infty) \to (0,\infty)$ which is decreasing in the first coordinate and increasing in the second, such that the following holds. Let $[\varphi] \in \mathcal{T}(S)$ and let $q(z)dz^2$ be a quadratic differential at $[\varphi]$. Let α be a gap with $\varphi(\partial\alpha)$ in good position, with $a(\alpha) = a$ and $|\varphi(\partial\alpha)|_q \leq M\sqrt{a}$. Let $J \subset \varphi(\alpha \cup \partial\alpha)$ be a segment of stable foliation with $|J|_q \geq \delta\sqrt{a}$. Then either every unstable segment in $\varphi(\alpha\cup\partial\alpha)$ of length $\geq L(\delta, M)\sqrt{a}$ intersects $J \cup \varphi(\partial\alpha)$ or there is a closed loop $\gamma \subset \mathrm{int}(\alpha)$ with $|\varphi(\gamma)|_q \leq L\sqrt{a}$ for some $L \leq L(\delta, M)$, and $|\varphi(\gamma)|_{q,-} \leq \varepsilon(L)\sqrt{a}$.*

Similar statements hold with the role of stable and unstable reversed.

Proof. By taking the oriented cover of the unstable foliation, we can assume without loss of generality that both the stable and unstable foliations are orientable. Then we fix a segment J of stable foliation with $|J|_q \geq \delta\sqrt{a}$. We write $J = \cup_{i=1}^p J_i$, and all unstable leaves starting from the interior of J_i in the positive direction either return to J_i without hitting singularities, or cross $\varphi(\alpha)$ without hitting singularities. Write R_j for the trapezium with base J_j formed in this manner. We number so that $|J_i|_q$ is decreasing in i. Then$|J_1|_q \geq \delta\sqrt{a}/p$ and hence all unstable segments in R_1 have unstable length $\leq (p/\delta + M)\sqrt{a}$ if the segments cross $\varphi(\partial\alpha)$ — using the bound $|\varphi(\partial\alpha)|_q \leq M\sqrt{a}$ and one segment being of qd-length $\leq p\sqrt{a}/\delta$. Similarly all unstable segments in R_j have unstable length $\leq a/|J_j|_q + M\sqrt{a}$. Write $x_j = |J_j|_q/\sqrt{a}$. Either $\sum_{k>j} x_k \geq \varepsilon(1/x_j + M)$ – in which case we also have $x_{j+1} \geq \varepsilon(1/x_j + 1/\delta)/p$ — or there is a least j such that $\sum_{k>j} x_k <$

$\varepsilon(1/x_j + M)$. In the first case we can write $g(x) = \varepsilon(1/x + M)/p$ – which is an increasing function of x – and we obtain $x_i \geq g^{i-1}(p/\delta + M)$ for all i, where g^{i-1} is the i-fold composition. In the second case we obtain this for $i \leq j$. So we obtain the result for $L(\delta, M) = 1/(g^{p-1}(p/\delta + M))$. □

Corollary 4.3. *Given $\delta > 0$, the following holds for suitable ltd parameter functions (Δ, r, s, K_0) and for a suitable function $L(\delta, \nu)$. Let α be a gap which is long ν-thick and dominant along a segment $\ell = [[\varphi_1], [\varphi_2]]$ and let $[\varphi] \in \ell$ with $d([\varphi], [\varphi_1]) \geq \Delta(\nu)$. Let $q(z)dz^2$ be the quadratic differential at $[\varphi]$ for $d([\varphi], [\varphi_2])$ with stable and unstable foliations $\mathcal{G}_\pm$. Let $a = a(\alpha, q)$.*

Then there is no segment of the unstable foliation of qd-length $\leq 2L(\nu, \delta)\sqrt{a}$ with both ends on $\varphi(\partial\alpha)$, and every segment of the unstable foliation of qd-length $\geq L(\nu, \delta)\sqrt{a}$ in $\varphi(\alpha)$ intersects every segment of stable foliation of length $\geq \delta\sqrt{a}$.

Similar statements hold with the role of stable and unstable reversed.

Proof. We apply the lemma with $\varepsilon(L) = C(\nu)L^{-1}$ for a suitable constant $C(\nu)$ relating the qd-metric and Poincaré metric, which ensures that if the second option $|\varphi(\gamma)|_q \leq L\sqrt{a}$ and $|\varphi(\gamma)|_{q,-} \leq \varepsilon(L)$ holds for L bounded in terms of $\Delta(\nu)$ and $\gamma \subset \alpha$ then there is $[\psi] \in [[\varphi_1], [\varphi]]$ with $|\psi(\gamma)| < \nu$, contradicting ν-thickness. Also if there is $\zeta \subset \alpha$ with endpoints on $\partial\alpha$ and not homotopic into the boundary such that $\varphi(\zeta)$ is a segment of unstable foliation and $|\varphi(\zeta)| \leq L(\nu, \delta)$, then adding in arcs along $\partial\alpha$ we again obtain a loop $\gamma \subset \alpha$ and $[\psi] \in [[\varphi_1], [\varphi]]$ with $|\psi(\gamma)| < \nu$, which again gives a contradiction. □

4.4. **Loops cut the surface into cells.** There are two fairly simple, but key, results, both of which follow directly from 4.2. These properties are used several times in [13], but may never be explicitly stated. The first may be reminiscent of the concept of tight geodesics in the curve complex developed by Masur and Minsky [5], and the point may be that these occur "naturally" in Teichmüller space.

Lemma *Given $L > 0$, there is a function $\Delta_1(\nu)$ depending only on the topological type of S, such that the following holds for suitable parameter functions (Δ, r, s, K_0) Let α be a gap which is long ν-thick and dominant along ℓ for (Δ, r, s, K_0), with $\Delta(\nu) \geq \Delta_1(\nu)$. Let $y_1 = [\varphi_1]$, $y_2 = [\varphi_2] \in \ell$ with $d(y_1, y_2) \geq \Delta_1(\nu)$. Let $\gamma_i \subset \alpha$ with $|\varphi_i(\gamma_i)| \leq L$, $i = 1, 2$.*

Then $\alpha \setminus (\gamma_1 \cup \gamma_2)$ is a union of topological discs with at most one puncture and topological annuli parallel to the boundary. Furthermore, for a constant $C_1 = C_1(L, \nu)$,

$$\#(\gamma_1 \cap \gamma_2) \geq C_1 \exp d(y_1, y_2).$$

Proof. Let $[\varphi]$ be the midpoint of $[[\varphi_1],[\varphi_2]]$, and let $q(z)dz^2$ be the quadratic differential for $d([\varphi],[\varphi_2])$ at $[\varphi]$. As before, write $a = a(\alpha)$. Because $|\psi(\gamma_i)| \geq \nu$ for all $[\psi] \in [[\varphi_1],[\varphi_2]]$, by 3.2, the good position of $\varphi_1(\gamma_1)$ satisfies

$$|\varphi_1(\gamma_1)|_{q,+} \geq C(L,\nu)\sqrt{a},$$

and similarly for $|\varphi_2(\gamma_2)|_{q,-}$. So

$$|\varphi(\gamma_1)|_{q,+} \geq C(L,\nu)e^{\Delta_1(\nu)/2}\sqrt{a},$$

$$|\varphi(\gamma_2)|_{q,-} \geq C(L,\nu)e^{\Delta_1(\nu)/2}\sqrt{a}.$$

Then 4.2 implies that, given ε, if $\Delta_1(\nu)$ is large enough given ε, $\varphi(\gamma_1)$ cuts every segment of stable foliation of $q(z)dz^2$ of qd-length $\geq \varepsilon\sqrt{a}$ and $\varphi(\gamma_2)$ cuts every segment of unstable foliation of $q(z)dz^2$ of qd- length $\geq \varepsilon\sqrt{a}$. So components of $\varphi(\alpha) \setminus (\varphi(\gamma_1) \cup \varphi(\gamma_2))$ have Poincaré diameter $< \nu$ if $\Delta_1(\nu)$ is sufficiently large, and must be topological discs with at most one puncture or boundary-parallel annuli.

The last statement also follows from 4.2. If $d(y_1,y_2) < \Delta_1(\nu)$, there is nothing to prove, so now assume that $d(y_1,y_2) \geq \Delta_1(\nu)$. It suffices to bound below the number of intersections of $\varphi(\gamma_1)$ and $\varphi(\gamma_1)$. Let $L(\nu,1)$ be as in 4.2, and assume without loss of generality that $L(\nu,1) \geq 1$. Supppose that $\Delta_1(\nu)$ is large enough that each of $\varphi(\gamma_1)$ and $\varphi(\gamma_2)$ contains at least one segment which is a qd-distance $\leq \sqrt{a}/L(\nu,1)$ from unstable and stable segments, respectively, of qd-length $\geq \sqrt{a}L(\nu,1)$. Note that the number of singularities of the quadratic differential is bounded in terms of the topological type of S. So apart from length which is a bounded multiple of $L(\nu,1)\sqrt{a}$, each of $\varphi(\gamma_1)$ and $\varphi(\gamma_2)$ is a union of such segments. Then applying 4.2, each such segment of $\varphi(\gamma_1)$ intersects each such segment on $\varphi(\gamma_2)$. So we obtain the result for $C_1 = c_0L(\nu,1)^{-2}$, for c_0 depending only on the topological type of S. □

4.5. **A partial order on ltd** (β,ℓ)**.** The following consequence of 4.2 is important for the implications of our main result, 3.7. It allows us to define a partial order on long thick and dominants (β,ℓ). The meaning of this, in terms of our "beads" metaphor, is that as adjacent beads representing (α_i,ℓ_i) and $(\alpha_{i+1},\ell_{i+1})$ of the sequence of 3.7 cannot slide past each other, no two distinct beads representing (α_i,ℓ_i) and (α_j,ℓ_j) can change positions on the string. (Of course, if they could do so, the metaphor would make no physical sense.)

Lemma *For $i = 1$, 3, let $y_i = [\psi_i] \in \ell_i$, and let β_i be a subsurface of S with $|\psi_i(\partial\beta_i)| \leq L$. Let ltd parameter functions be suitably chosen given L. Let $\ell_2 \subset [y_1,y_3]$ and let $\beta_2 \cap \beta_i \neq \emptyset$ for both $i = 1$, 3, and let β_2 be ltd along ℓ_2. Then $\beta_1 \cap \beta_3 \neq \emptyset$, and β_2 is in the convex hull of β_1 and β_3.*

Proof. This is obvious unless both $\partial\beta_1$ and $\partial\beta_3$ intersect the interior of β_2. So now suppose that they both do this. First suppose that β_2 is a gap and long, ν-thick and dominant. Let $y_{2,1} = [\psi_{2,1}]$, $y_{2,3} = [\psi_{2,3}] \in \ell_2$ with $y_{2,i}$ separating ℓ_i from y_2, with $y_{2,i}$ distance $\geq \frac{1}{3}\Delta(\nu)$ from the ends of ℓ_2 and from y_2. If β_2 is a loop, then we can take these distances to be $\geq \frac{1}{6}\log K_0$. For $[\psi] \in [y_-, y_+]$, let $\psi(\beta)$ denote the region bounded by $\psi(\partial\beta)$ and homotopic to $\psi(\beta)$, assuming $\psi(\partial\beta)$ is in good position with respect to the quadratic differential at $[\psi]$ for $[y_-, y_+]$. Then if β_2 is a gap, $\psi_{2,1}(\partial\beta_1 \cap \beta_2)$ includes a union of segments in approximately unstable direction, of Poincaré length bounded from 0, and similarly for $\psi_{2,3}(\partial\beta_3 \cap \beta_2)$, with unstable replaced by stable. Then as in 4.4, $\psi_2(\partial\beta_3 \cap \beta_2)$ and $\psi_2(\partial\beta_1 \cap \beta_2)$ cut $\psi_2(\beta_2)$ into topological discs with at most one puncture and annuli parallel to the boundary. It follows that β_2 is contained in the convex hull of β_1 and β_3. If β_2 is a loop, it is simpler. We replace $\psi(\beta_2)$ by the maximal flat annulus $S([\psi])$ homotopic to $\psi(\beta_2)$, for $[\psi] \in \ell_2$. Then $\psi_2(\partial\beta_1) \cap S([\psi_2])$ is in approximately the unstable direction and $\psi_2(\partial\beta_3) \cap S([\psi_2])$ in approximately the stable direction. They both cross $S([\psi_2])$, so must intersect in a loop homotopic to $\psi_2(\beta_2)$. □

We define $(\beta_1, \ell_1) < (\beta_2, \ell_2)$ if ℓ_1 is to the left of ℓ_2 (in some common geodesic segment) and $\beta_1 \cap \beta_2 \neq \emptyset$. We can make this definition for any segments in a larger common geodesic segment, and even for single points in a common geodesic segment. So in the same way we can define $(\beta_1, y_1) < (\beta_2, \ell_2)$ if y_1 is to the left of ℓ_2, still with $\beta_1 \cap \beta_2 \neq \emptyset$, and so on. This ordering is transitive restricted to ltd's (β_i, ℓ_i) by the lemma.

4.6. **The graph of the qd-length function.** One of the basic technical considerations in the study of Teichmüller geodesics, as is probably already apparent, is the difference between the qd- and Poincaré metrics. The two metrics are not globally Lipschitz equivalent. But they are Lipschitz equivalent, up to scalar, on any thick part of a surface. The Lipschitz constant is bounded in terms of the topological type of the surface, but the scalar is completely uncontrollable. This should not be regarded as a problem. One simply has to look at ratios of lengths rather than at absolute lengths.

We consider a fixed geodesic segment $[y_0, y_T]$, as in 4.1 and 3.7. We use the notation of 4.1, so that $y_t = [\varphi_t] = [\chi_t \circ \varphi_0]$, and $q_t(z)dz^2$ is the quadratic differential at y_t for $d(y_0, y_t)$: the stretch of $q_0(z)dz^2$ at y_0 (see 2.3). For any finite loop set γ, we define the *qd-length function for* γ by

$$F(t, \gamma) = \log |\varphi_t(\gamma)|_t.$$

By 14.7 of [13] (and I am sure this is well-known), this function has a remarkable property. There is $t(\gamma) \in \mathbb{R}$, and a constant C_0 depending only

on the topological type of S and a bound on the number of loops in γ, such that

$$|F(t,\gamma)-F(t(\gamma),\gamma)-|t-t(\gamma)||\le C_0. \tag{4.6.1}$$

As a consequence of this, and of the theory of section 3, we have the following. As already noted in 4.1, this will not be sufficient for our purposes, but it is a start, and is, in fact, used in the construction of α_1.

Lemma 4.7. *Fix a topological surface S and ltd parameter functions (Δ, r, s, K_0) and $\nu_0>0$. The following holds for $C_1>0$ depending only on the topological type of S, and constants s_0 and L depending on the topological type of S and (Δ, r, s, K_0) and ν_0, where L is as in 3.5, and for any sufficiently large $\Delta_1>0$ given these. Let $[y_0,y_T]=[[\varphi_0],[\varphi_T]]=\{[\varphi_t]: t\in[0,T]\}$ be a Teichmüller geodesic segment, and $\ell\subset[y_0,y_T]$ a segment of length $\ge\Delta_1$. Let β be disjoint from all subsurfaces α of S such that (α,ℓ') is ltd with respect to $(\Delta, r, s, K_0, \nu_0)$ for some $\ell'\subset\ell$. Then for all $t\in[0,T]$,*

$$s_0a'(\beta)\le|\varphi_t(\partial\beta)|_t^2, \tag{4.7.1}$$

and hence

$$a'(\beta)\le C_1s_0^{-1}L^2e^{-\Delta_1}. \tag{4.7.2}$$

Proof. Enlarge β if necessary, so that $\partial\beta$ is contained in the convex hull of the union of $\partial\alpha$ such that (α,ℓ') is ltd and $\ell'\subset\ell$. By 3.5, for $[\varphi]\in\ell$:

$$|\varphi(\partial\beta)|\le L. \tag{4.7.3}$$

So if $\ell\subset[y_0,y_T]$, the function $F(t,\partial\beta)=|\varphi_t(\partial\beta)|_t$ is bounded, in terms of L, on an interval of length $\ge\Delta_1$. There is also a constant C_0 depending only on the topological type of S such that

$$|\varphi(\gamma)|_q\le C_0|\varphi(\gamma)|$$

for any closed loop γ on S and $[\varphi]\in\mathcal{T}(S)$ and quadratic differential $q(z)dz^2$ at $[\varphi]$. We apply this with $\varphi=\varphi_t$ and $q=q_t$ for varying t. So by (4.6.1), there is at least one t such that

$$|\varphi_t(\partial\beta)|_t\le C_1e^{-\Delta_1/2}L$$

Let $s_0=s_0(\Delta,r,s,K_0)>0$ as in 3.4. By the last part of 3.4, we have

$$e^{2F(t,\partial\beta)}=|\varphi_t(\partial\beta)|_t^2\ge s_0a'(\beta), \tag{4.7.4}$$

which gives (4.7.1). But $a'(\beta)$ is independent of t. So (4.7.2) follows, with $C_1=C_0^2$. □

4.8. **Comparision between Poincaré length and qd-length.** Comparision between Poincaré and qd-length can be made as follows. Given $L_1 > 0$ there is $L_2 \in \mathbb{R}$ such that if

$$|\varphi_t(\gamma)| \leq L_1 \tag{4.8.1}$$

then

$$F(t,\gamma) - F(t,\gamma') \leq L_2 \tag{4.8.2}$$

for all nontrivial nonperipheral γ' intersecting γ transversely. Conversely, given $L_2 \in \mathbb{R}$, there is L_1 such that (4.8.1) holds whenever (4.8.2) holds for all γ' intersecting γ transversely. There is a similar characterisation of short loops. Given $L_2 < 0$, there is $L_1 > 0$ (which is small if L_2 is negatively large) such that, whenever (4.8.1) holds, then (4.8.2) holds for all γ' intersecting γ tranversely. Conversely, given $L_1 > 0$, there is L_2 (which is negative if L_1 is small) such that (4.8.1) holds for γ, whenever (4.8.2) holds for γ and all γ' transverse to γ.

We now start to deal with the problem we identified in 4.1: given t, how to bound the intersection of $\varphi_t(\beta)$ with a segment ζ_1 of unstable foliation of bounded Poincaré length, for varying t. In the following series of lemmas 4.9 - 4.12, we split up the segments of unstable foliation across $\varphi_t(\beta)$ into sets which are dealt with separately, some of them encased in larger subsurfaces. We then have to make estimates on these larger subsurfaces, and also on the number of them.

Lemma 4.9. *Let γ be an arc such that $\varphi_t(\gamma)$ is a segment of unstable foliation for one, and hence all, t. Given L there is L_1 such that if $\varphi_t(\gamma)$ has Poincaré length $\leq L$ then $\varphi_u(\gamma)$ has Poincaré length $\leq L_1$ for all $u \leq t$. Moreover, if ε_0 is any Margulis constant then given $C > 0$ there exists $C_1 > 0$ such that, if $\gamma \subset (\varphi_t(S))_{<\varepsilon_0}$ or $\gamma \subset (\varphi_t(S))_{\geq\varepsilon_0}$, and $\varphi_t(\gamma)$ has Poincaré length $\leq C$ times the injectivity radius, then $\varphi_u(\gamma)$ has Poincaré length $\leq C_1$ times the injectivity radius, for all $u \leq t$.*

Similar statements hold for stable segments and $u \geq t$.

Proof. Since a segment of length $\leq L$ can be split up into a number of segments of length bounded by the injectivity radius, where the number is bounded in terms of L, the first statement follows from the second. So take any such segment γ of length bounded by the injectivity radius at a point on γ, and take any closed loop ζ intersecting γ. Then $\log|\varphi_u(\gamma)|_u - \log|\varphi_u(\zeta)|_u$ is non-increasing for $u \leq t$. The result follows. $\square$

Lemma 4.10. *Given $\eta_1 > 0$ and $L > 0$ there exists $\eta_2 > 0$ depending only on η_1, L and the topological type of S, such that the following holds for any sufficiently large $\Delta_1 > 0$. Let β be a gap or loop with $|\varphi_t(\partial\beta)| \leq L$*

and $a'(\beta) \leq e^{-\Delta_1}$. *Then there is* $\beta' \supset \beta$ *such that* $a'(\beta') \leq e^{-(1-\eta_1)\Delta_1}$ *and* $|\varphi_u(\partial\beta')| \leq e^{-\eta_2\Delta_1}$ *for* $|u-t| \leq \eta_2\Delta_1$.

Proof. Choose $\varepsilon_0 > 0$ such that no loop of Poincaré length $\leq \varepsilon_0$ is intersected transversely by a loop of length $\leq L$. For any $\eta_3 > 0$ sufficiently small depending only on the topological type of S we can find a connected union β' of gaps and loops ω containing β such that $|\varphi_t(\partial\omega)| \leq \varepsilon_0$, and if ω' is adjacent to ω and also in this union then $e^{-\eta_3\Delta_1} \leq a'(\omega')/a'(\omega) \leq e^{\eta_3\Delta_1}$, but if ω is inside the union and ω' outside then $a'(\omega') > e^{\eta_3\Delta_1}a'(\omega)$. It follows that for any such ω and ω' we have

$$|\varphi_t(\partial\omega' \cap \partial\omega)| \leq e^{-\eta_3\Delta_1/2}\mathrm{Max}(a'(\omega), a'(\omega')).$$

So then for $|u-t| \leq \eta_3\Delta_1/4$ we have

$$|\varphi_u(\partial\omega' \cap \partial\omega)| \leq e^{-\eta_3\Delta_1/4}\mathrm{Max}(a'(\omega), a'(\omega')).$$

Now there are at most p sets ω in the union, where p depends only on the topological type of S and hence

$$a'(\beta') \leq e^{p\eta_3\Delta_1}a'(\beta) \leq e^{-(1-p\eta_3)\Delta_1}.$$

So if we choose η_3 with $p\eta_3 \leq \eta_1$ we have $a'(\beta') \leq e^{-(1-\eta_1)\Delta_1}$ and the bound on $|\varphi_u(\partial\beta')|$ holds for $\eta_2 = \eta_3/4$. □

Lemma 4.11. *Let* $|\varphi_{u_1}(\partial\beta)| \leq L$ *and* $|\varphi_{u_1}(\partial\beta)|_{u_1,-} \geq \frac{1}{2}|\varphi_{u_1}(\partial\beta)|_{u_1}$, *and let* $a'(\beta) \leq e^{-\Delta_1}$. *Then if* Δ_1 *is sufficiently large given constants* μ_i, L *and the topological type of* S, *there is an increasing sequence of subsurfaces* $(\beta_i : 1 \leq i \leq k)$ *with* $\beta = \beta_1$, *sequences of surfaces* $(\omega_i = \omega_i(\beta) : 1 \leq i \leq k)$ *and* $(\omega_i' = \omega_i'(\beta) : 1 \leq i \leq k-1)$ *with* $\omega_1 = \beta$, $\omega_k = \emptyset$, *a decreasing sequence of real numbers* $u_i = u_i(\beta)$, *and constants* L_i *depending only on* L *and* μ_j *for* $j < i$ *and the topological type of* S, *such that the following hold.*

- $\beta_i \setminus \beta_{i-1} \subset \omega_i \subset \beta_i$ *and* $\omega_i' \subset \omega_i \cap \omega_{i+1}$, *so that* $\beta_k = \cup_{i=1}^{k-1}\omega_i \setminus \omega_{i+1}$.
- $|\varphi_t(\partial\omega_i)| \leq L_i$ *for* $u_i \leq t \leq u_{i-1}$, *for* $2 \leq i \leq k$, $|\varphi_{u_i}(\gamma)|_{u_i,-} \geq \frac{1}{2}|\varphi_{u_i}(\gamma)|_{u_i}$ *for some component* γ *of* $\partial\beta_i$ *with* $|\varphi_{u_i}(\gamma)| \geq 1$.
- *For all* $t \leq u_i$, *every segment* ζ *of unstable foliation across* $\varphi_t(\omega_i \setminus \omega_i')$ *has Poincaré length* $\leq L_i$ *for all* $t \leq u_i$, *and every maximal segment* ζ *of unstable foliation across* $\varphi_t(\omega_i \setminus \omega_{i+1})$ *is adjacent on each side to a segment* ζ' *of unstable foliation in* $\varphi_t(S \setminus \omega_i)$ *with* $|\zeta|_t \leq \mu_i|\zeta'|_t$.
- $a'(\beta_{i+1}) \leq CL_i\mu_i^{-1}a'(\beta_i)$, *where* C *depends only on the topological type of* S.
- *Either* β_{i+1} *is strictly bigger than* β_i *or* ω_{i+1} *is strictly smaller than* ω_i.
- *Either* $u_k = 0$ *or every unstable segment across* $\varphi_{u_k}(\beta_k)$ *has Poincaré length* $\leq L_k$ *and every unstable segment* ζ *across* $\varphi_{u_k}(\beta_k)$ *is adjacent to a segment* ζ' *of* $\varphi_{u_k}(S \setminus \beta_k)$ *with* $|\zeta| \leq \mu_k|\zeta'|$.

Remark Note that there is no claim that $|\varphi_{u_i}(\beta_{i-1})|$ is bounded.

Proof. We start the inductive definitions with $\beta = \beta_1 = \omega_1$. By the Fundamental Lemma 4.2, for a constant L_1 depending only on L and the topological type of S, there is a subsurface ω_1' of β — which could be empty — such that $|\varphi_{u_1}(\partial\omega_1')| \leq L_1$ and

$$|\varphi_{u_1}(\partial\omega_1')|_{u_1,-} \leq L^{-1}|\varphi_{u_1}(\partial\omega_1')|_{u_1}.$$

and every unstable segment across $\varphi_{u_1}(\beta \setminus \omega_1')$ has Poincaré length $\leq L_1$. The same estimate for $u \leq u_1$ follows by 4.9, for L_1 sufficiently large given L.

Let β_2' be the surface which is the union of $\beta = \beta_1$ and every maximal unstable segment ζ' outside β which is adjacent to an unstable segment ζ across $\varphi_{u_1}(\beta\setminus\omega_1')$, such that $|\zeta'|_{u_1} < \mu^{-1}|\zeta|_{u_1}$. Then $a'(\beta_2') \leq (1+\mu^{-1})a'(\beta)$. Now let β_2 be the surface containing β_2' such that $(\beta_2 \setminus \beta_2') \cup (\beta_2' \setminus \beta_2)$ is a union of discs and annuli, and such that $\varphi_{u_1}(\partial\beta_2)$ is in good position. Let $\varphi_{u_1}(\omega_2) \subset \varphi_{u_1}(\beta_2)$ be the union of $\varphi_{u_1}(\omega_1')$ and the surface obtained by leaving out those unstable segments ζ in $\varphi_{u_1}(\beta)$ for which an adjacent unstable segment $\zeta' \subset \varphi_{u_1}(S \setminus \beta)$ exists on each side, with $|\zeta|_{u_1} \leq \mu|\zeta'|_{u_1}$, again homotoping so that $\varphi_{u_1}(\omega_2)$ is in good position. Thus $\varphi_{u_1}(\omega_2)$ is obtained from $\varphi_{u_1}(\beta_2)$ by leaving out some handles with boundary which was in $\varphi_{u_1}(\partial\beta)$, before the good position homotopy. Then $|\varphi_{u_1}(\omega_2)| \leq \mu^{-1}L_1$, assuming that $CL \leq L_1$, for a suitable constant C depending only on the topological type of S.

Now we prove that $a'(\beta_2) \leq C\mu^{-1}a'(\beta)$, again assuming that C is suitably chosen. The surface $\varphi_{u_1}(\beta_2 \setminus \beta_2')$ is a union of topological discs and annuli. So the aim is to bound the areas of the added discs and annuli. Each topological disc is a polygon with each side at a constant slope to the stable and unstable foliation, with alternate sides tangent to the unstable foliation. The number of sides of each polygon is bounded by the number and type of singularities inside the polygon, which is bounded by the topological type of S. So the total number of polygons in $\varphi_{u_1}(\beta_2 \setminus \beta_2')$ which have more than four sides is bounded by the topological type of S.

The area of any polygon with at most four sides – a trapezium – is bounded by the product of the length of two adjacent sides, that is, by μ^{-1} times the product of the lengths of an adjacent triangle or trapezium in $\varphi_{u_1}(\beta)$. For any of the other boundedly finitely many polygons, we foliate by unstable segments, and so obtain the polygon as a finite union of trapezia, such that only unstable sides of trapezia can be in the interior of the polygon, and sides which are not unstable segments are in $\varphi_{u_1}(\partial\beta)$. So then, by induction, we obtain a bound on the area of the polygon in terms

of μ^{-1} times the area of adjacent trapezia in $\varphi_{u_1}(\beta)$. So the area of the polygon is $\leq C_1\mu^{-1}a'(\beta)$ for a suitable constant C_1.

To obtain a similar area bound for the annuli in $\varphi_{u_1}(\beta_2 \setminus \beta_2')$, it suffices to bound the number of polygons in an annulus $\varphi_{u_1}(A)$ bounded by $\varphi_{u_1}(\gamma)$ and $\varphi_{u_1}(\gamma')$, where γ and γ' are homotopic components of $\partial\beta_2$ and $\partial\beta_2'$ respectively. Since $\varphi_{u_1}(\gamma)$ is in good position, we have a bound, in terms of the topological type of S, on the number of constant slope segments on $\varphi_{u_1}(\gamma)$. We need a bound on the number of constant slope segments on $\varphi_{u_1}(\gamma'$, which means bounding the number of segments of $\varphi_{u_1}(\partial\beta)$ and $\varphi_{u_1}(\partial\beta_2')$ on $\varphi_{u_1}(\gamma')$. To do this, we consider the trapezia in $\varphi_{u_1}(\beta_2'\setminus\beta)$ which are subsets of the boundedly finitely many trapezia in $\varphi_{u_1}(S\setminus\beta)$. Both sets of trapezia are foliated by unstable segments and have their other sides in $\varphi_{u_1}(\partial\beta)$. No two trapezia of $\varphi_{u_1}(\beta_2'\setminus\beta)$ can be in the same trapezium of $\varphi_{u_1}(S\setminus\beta)$ and bounded by the same trapezia of $\varphi_{u_1}(\beta)$. So the number of trapezia in $\varphi_{u_1}(\beta_2'\setminus\beta)$ is bounded. So the number of boundary components of these trapezia is bounded, and hence the number that can intersect $\varphi_{u_1}(\gamma')$ is bounded.

If $\beta_2 = \beta_1$ and $\omega_2 = \omega_1' = \emptyset$, then we define $k = 1$. Now suppose that $\beta_2 \neq \beta_1$ and $\omega_2 \neq \emptyset$. Then choose the first u_2 to be the first $t \leq u_1$ such that $|\varphi_t(\gamma)|_{t,-} = \frac{1}{2}|\varphi_t(\gamma)|_t$ for a component γ of $\partial\beta_2$ with $|\varphi_t(\gamma)| \geq 1$. Then, from the bound on $|\varphi_{u_1}(\partial\omega_1'\cup\partial\beta)|$, we have $|\varphi_t(\partial\omega_2)| \leq L_2$ for $u_2 \leq t \leq u_1$, for L_2 depending only on L and μ_1 . By the Fundamental Lemma 4.2, enlarging L_2 if necessary, but still depending only on L and μ_1, there is $\omega_2' \subset \omega_2$ (where ω_2' is allowed to be empty) such that $|\varphi_{u_2}(\partial\omega_2')| \leq L_2$, and $|\zeta| \leq L_2$ for every unstable segment ζ across $\varphi_{u_2}(\omega_2 \setminus \omega_2')$ has Poincaré length $\leq L_2$ and

$$|\varphi_{u_2}(\partial\omega_2')|_{u_2,-} \leq L^{-1}|\varphi_{u_2}(\partial\omega_2)|_{u_2}$$

We define $\varphi_{u_2}(\beta_3')$ to be the union of $\varphi_{u_2}(\beta_2)$ and of all unstable segments ζ' in $\varphi_{u_2}(S\setminus\omega_2)$ such that ζ' has both endpoints in $\varphi_{u_2}(\partial\omega_2)$ and ζ' is adjacent to an unstable segment ζ in $\varphi_{u_2}(\omega_2 \setminus \omega_2')$ with $|\zeta'| \leq \mu_2^{-1}|\zeta|$. We therefore have $a'(\beta_3') \leq \mu_2^{-1}a'(\beta_2)$. We then define β_3, ω_3 and L_3 from β_3', ω_2', L_2 and μ_2 in exactly the same way as β_2, ω_2 and L_2 are defined from β_2', L and μ, and continue to define u_3 and ω_3' analogously to u_2 and ω_2'. The definition of β_i, ω_i, ω_i', L_i and u_i is exactly the same for all $i \geq 3$. The bound on $a'(\beta_i)$ for $i \geq 3$ works in the same way as the bound on $a'(\beta_2)$.

Since any sequence of subsurfaces of S of strictly increasing or strictly decreasing topological type is bounded – in terms of the topological type of S, there is k bounded in terms of the topological type of S such that $\beta_k = \beta_{k-1}$ and $\omega_k = \emptyset$ with $\beta \subset \beta_k$ and $a'(\beta_k) \leq \prod_{i\leq k} L_i\mu_i^{-1}a'(\beta)$. If Δ_1 is sufficiently large, it follows that $\beta_k \neq S$. $\square$

For the sets $\beta_j = \beta_j(\beta)$ and $\omega_j = \omega_j(\beta)$ as in 4.11, we have

$$\beta_{i-1} \setminus \omega_i = \cup_{j=1}^{i-1}(\omega_{j+1} \setminus \omega_j).$$

Lemma 4.12. *Fix a Teichmuller geodesic segment $\ell_0 = [y_{s_1}, y_{s_0}]$ with $|\ell_0| \leq p_1\Delta_1$. The number of β such that $|\varphi_t(\partial\beta)| \leq L$ for t in an interval of $[s_1, s_0]$ of length $\geq \Delta_1/p_1$ is bounded in terms of L, p_1 and the topological type of S. Let $u_i(\beta)$ and $\omega_i(\beta)$ be as in 4.11. The number of $\omega_i = \omega_i(\beta)$ with $y_{u_i} \in \ell_0$ is bounded in terms of L, p_1, $(\mu_j : j < i)$ and the topological type of S, in any interval of length Δ_1, for any $i \leq k = k(\beta)$, even if $y_{u_1} \notin \ell_0$.*

Proof. The loop set $\partial\beta$ is a *multicurve* (see 2.1), that is, a set of simple closed loops which are homotopically disjoint and distinct. For any fixed t, the number of multicurves in $\varphi_t(S)$ of length $\leq L$ is $\leq C_1L^{6g-6+2b}$ where g is the genus of S and b the number of boundary components and C_1 is a universal constant – just depending on the Margulis constant in two dimensions. So by considering $p_1^2 + 1$ points in ℓ_0 such that any other point of ℓ_0 is distance $\leq \Delta_1$ from one of these, we see that the number of choices for β is $\leq L^{(p_1^2+1)(6g-6+2b)}$ if $y_{t_1} \in \ell$. Now suppose that $y_{u_i} \in \ell_0$. If $y_{u_1} \notin \ell_0$ then $s_0 \in [u_j, u_{j-1}]$ for some $j \leq i$. Since $|\varphi_t(\partial\omega_j)| \leq L_j$ for $t \in [u_j, u_{j-1}]$, this is true for $t = s_0$, and hence the number of choices for this ω_j is $\leq C_1L_j^{6g-6+2b}$, where L_j depends only on L and $\mu_{j'}$ for $j' < j$. Then u_j is determined from ω_j to within a bounded distance by the property $|\varphi_{u_j}(\gamma)|_{u_j,-} \geq \frac{1}{2}|\varphi_{u_j}(\gamma)|_{u_j}$ for a component γ of $\partial\omega_j$ with $|\varphi_{u_j}(\gamma)| \geq 1$. We also have $|\varphi_{u_j}(\partial\omega_{j+1})| \leq L_j$. So the number of choices for ω_{j+1}, given u_j, is also $\leq C_1L_j^{6g-6+2b}$. Then from ω_{j+1} and the predetermined L_{j+1} we can determine u_{j+1}, and hence we have a bound on the number of choices for $\omega_{i'}$ for all $i' \leq i$ which depends only on p_1, $\mu_{i'}$ for $i' < i$ and the topological type of S.

□

We now have the estimates in place to bound the intersection of unstable segments with "bad set", that is, the set of β bounded by L. But there is still some work to do on the complement, the "good set" because this is the convex hull of long thick and dominants, rather than their union. So we need the following.

Lemma 4.13. *Fix ltd parameter functions (Δ, r, s, K_0) and ν_0 as in 3.4. There is a constant M depending on these such that the following holds. Let $\ell = [y_{u_1}, y_{u_0}] \subset [y_0, y_t]$ be any Teichmüller geodesic segment such that there is at least one ltd (α, ℓ') with $\ell' \subset \ell$. Let Ω be the convex hull of all such α and let Ω' be the union of all such α. Then any unstable segment ζ of $\varphi_t(\Omega \setminus \Omega')$ is adjacent to a unstable segment ζ' in $\varphi_t(\alpha)$ for some ltd (α, ℓ') with $\ell' \subset \ell$ with $|\zeta'|_t \geq |\zeta|_t/M$ for all t. A similar statement holds for stable segments.*

Proof. We treat the case of unstable segments. The proof for stable segments is exactly analogous. There are finitely may ltd's $(\alpha_i, [y_{w_i}, y_{v_i}])$ with $1 \leq i \leq k$ and $w_i \leq w_{i+1}$ and $u_1 \leq w_i$ for all i, such that Ω_i is of larger topological type than Ω_{i-1}, where Ω_i is the convex hull of $\cup_{j \leq i} \alpha_j$ and $\Omega_0 = \emptyset$, and there is no ltd (α, ℓ) such that α has essential intersection with $\Omega_i \setminus \Omega_{i-1}$ and $\ell \subset [y_{u_1}, y_{w_i}]$. We say that α_i is *visible from* y_{u_1}, meaning that parts of it are. The visibility property means that, by 3.5, there is a constant L_1 depending only on $(\Delta, r, s, K_0, \nu_0)$ and the topological type of S such that $|\varphi_u(\partial\Omega_i)| \leq L_1$ for $u_1 \leq u \leq \max(v_i, w_{i+1})$, for each i.

We then aim to show inductively that, for a constant $c > 0$ depending only on $(\Delta, r, s, K_0, \nu_0)$ and the topological type of S, and any unstable segment ζ across $\varphi_u(\Omega_i)$,

$$|\zeta|_u \geq c|\zeta \cap (\cup_{j \leq i} \varphi_u(\alpha_j))|_u \tag{4.13.1}$$

for all u, and all i, and, if γ is a component of $\partial\Omega_i$ such that $\varphi_u(\gamma)$ contains an endpoint of ζ, then, if $u \geq w_i$,

$$|\varphi_u(\gamma)|_u \geq c|\zeta|_u. \tag{4.13.2}$$

These statements suffice, because γ is a union of boundedly finitely many segments in the sets α_j for $j \leq i$, where, for each j, each segment of $\gamma \cap \alpha_j$ is disjoint from all ltd's (α', ℓ') with $\ell' \subset [y_{u_1}, y_{w_j}]$. In both cases, if the inequalities hold for $u = w_i$, they hold for all claimed u. For (4.13.1), this is because for different u, the two sides of the inequality are scaled by the same factor. For (4.13.2), the lefthand side of the inequality is obtained from that for $u = w_i$ by multiplying by e^{u-w_i} and the right-hand side is obtained by mutliplying by at most e^{u-w_i}. The statements are trivially true for $\Omega_1 = \alpha_1$, by considering $u = w_1$. So we consider the inductive statements. We assume they are true for Ω_{i-1} with $i \geq 2$. Now $\varphi_u(\Omega_i)$ is obtained from $\varphi_u(\Omega_{i-1} \cup \alpha_i)$ by adding annuli and topological discs, each of which is bounded by transversally intersecting components of $\varphi_u(\partial\Omega_{i-1})$ and $\varphi_u(\partial\alpha_i)$. For $u = w_i$, if γ and γ' are transversally intersecting components of $\partial\Omega_{i-1}$ and $\partial\alpha_i$, then $|\varphi_u(\gamma)|$ and $|\varphi_u(\gamma')|$ (that is, the Poincaré lengths) are boundedly proportional, with bound depending on the ltd parameter functions. Therefore $|\varphi_{w_i}(\gamma)|_{w_i}$ and $|\varphi_{w_i}(\gamma')|_{w_i}$ are also boundedly proportional. So (4.13.2) is true by induction — possibly after modfifying c, but since this only has to be done for each i and the number of i is bounded in terms of the topological type of S, this is allowed. Unstable segments across any added topological disc or annulus have qd-length bounded by a constant times $|\varphi_{w_i}(\gamma)|_{w_i}$ for any component γ of $\partial\Omega_{i-1}$ or $\partial\alpha_i$ such that $\varphi_{w_i}(\gamma)$ intersects the boundary of this disc or annulus. This is less than $C_1|\zeta|_{w_i}$ for any adjacent unstable segment ζ in $\varphi_{w_i}(\Omega_{i-1})$, by (4.13.2) for $i-1$ replacing

i, and for any unstable segment ζ in $\varphi_{w_i}(\alpha_i)$, since $(\alpha_i[y_{w_i}, y_{v_i}])$ is ltd. So we obtain

$$|\zeta|_{w_i} \geq c_1|\zeta \cap \varphi_{w_i}(\Omega_{i-1} \cup \alpha_i)|_{w_i},$$

for a suitable $C_1 > 0$, and then (4.13.1) also follows, by induction.

□

4.14. **Proof of 3.7: construction of the sequences.** For $1 \leq i \leq R_0$, some R_0, we shall find sequences t_i, α_i, ζ_i, such that the following hold. We start by choosing t_1 with $0 \leq t_1 \leq \Delta_0/2$. But after that, to simplify the writing, we assume that $t_1 = 0$. In the most technical property, 5, the constant μ is as in 4.11, and p is any integer such that $1/p \leq \eta_2/3$, for η_2 as in 4.10. For any suitable μ and p, property 5 will be obtained if Δ_0 is large enough.

1. (α_i, ℓ_i) is ltd at $y_{t_i} \in \ell_i \subset [y_0, y_T]$ with respect to (Δ, r, s, K_0), in the first quarter of ℓ_i if α_i is a loop, and $a'(\alpha_i) \geq e^{-\Delta_0/2}$.
2. $\zeta_{i+1} \subset \zeta_i$ and $\zeta_i \subset \varphi_0(\alpha_i)$ if α_i is a gap, and $\zeta_i \subset \varphi_0(A(\alpha_i))$ if α_i is a loop, where $\varphi_t(A(\alpha_i))$ is the flat annulus in the q_t-metric which is homotopic to $\varphi_t(\alpha)$.
3. Writing $t_1 = 0$, for all $i \geq 1$, $\chi_{t_i}(\zeta_i)$ is a segment of the unstable foliation of the quadratic differential $q_{t_i}(z)dz^2$ whose Poincaré length is boundedly proportional to the injectivity radius at that point of $\varphi_{t_i}(S)$.
4. $t_1 \leq \Delta_0/2$ and $t_{R_0} \geq T - \Delta_0$. For all $1 \leq i < R_0$, $t_i < t_{i+1} \leq t_i + \Delta_0$.
5. Each segment $\varphi_{t_i}(\gamma)$ of $\chi_{t_i}(\zeta_i) \cap \varphi_{t_i}(\omega_j(\beta) \setminus \omega_{j+1}(\beta))$ is adjacent to a segment of $\chi_{t_i}(\zeta_i) \setminus (\varphi_{t_i}(\omega_j(\beta) \setminus \omega_{j+1}(\beta)))$ whose (Poincaré or qd) length is at least $\mu_j^{-1/2}$ times more, for any β with $a'(\beta) \leq e^{-\Delta_0/3}$ and $|\varphi_t(\partial\beta)| \leq L$ for $|t - u_0| \leq \Delta_0/p$ and $u_0 = u_0(\beta) \geq t_i + \Delta_0$. If $|\varphi_{t_i}(\gamma)|_{t_i} \geq \mu^{-1}|\chi_{t_i}(\zeta_i)|_{t_i}$ for such a γ, then $\varphi_t(\gamma) \cap \chi_t(\zeta_{i+1}) = \emptyset$ for all t.

1,2 and 4 give the proof of 3.7, apart from $a'(\alpha_i) > \delta_0$, since, by 2, we have

$$\zeta_j \subset \varphi_{t_1}(\alpha_i)$$

for all $i \leq j$. 3 and 5 are needed for the inductive process. The notation of Property 5 comes from 4.11.

Note that property 5 implies that $a'(\alpha_i) > e^{-5\Delta_0/14}$, which gives $a'(\alpha_i) > \delta_0$, if $\delta_0 = e^{-5\Delta_0/14}$. For if $a'(\alpha_i) \leq e^{-5\Delta_0/14}$, then by 4.10 there is $\beta \supset \alpha$ with $|\varphi_t(\partial\beta)| \leq e^{-\Delta_0/p}$ for $|t - t_i| \leq \Delta_0/p$ and $a'(\beta) < e^{-\Delta_0/3}$, for a suitably chosen p depending only on the topological type of S. But then by Property 5 for this β, and the properties of unstable segment across $\varphi_t(\omega_j \setminus \omega_{j+1})$ of 4.11, ζ_i is not contained in $\cup_j \varphi_0(\omega_j(\beta) \setminus \omega_{j+1}(\beta))$, assuming the μ_j grow

sufficiently fast. But since $\beta \subset \cup_j \omega_j(\beta) \backslash \omega_{j+1}(\beta)$, this gives the contradiction that α is not contained in β. So we do have $a'(\alpha_i) > e^{-5\Delta_0/14}$. Then since $\chi_{t_i}(\zeta_i)$ has Poincaré length bounded from 0 if α_i is a gap and boundedly proportional to the injectivity radius if α_i is a loop, we obtain $|\chi_{t_i}(\zeta_i)|_{t_i} \geq e^{-5\Delta_0/29}$ in both cases, provided that, when α_i is a loop, we choose t_i such that y_{t_i} is towards the right end of the segment ℓ_i along which (α_i, ℓ_i) is ltd, which we can do by inserting extra points t_j with $j < i$ $y_{t_j} \in \ell_i = \ell_j$ and $\alpha_j = \alpha_i$, if necessary. Now if t is chosen so that $t_i + \Delta_0/2 \leq t \leq t_i + \Delta_0$, and $\zeta \subset \zeta_i$ is any segment such that $|\chi_t(\zeta)|$ is bounded, then $|\chi_t(\zeta)|_t$ is also bounded, and $|\chi_{t_i}(\zeta)|_{t_i} \leq e^{-\Delta_0/2}|\chi_t(\zeta)|_t$. So

$$|\chi_{t_i}(\zeta)|_{t_i}/|\chi_{t_i}(\zeta_i)|_{t_i} \leq e^{-\Delta_0/4}. \tag{4.14.1}$$

In particular, (4.14.1) holds if $t_i + \Delta_0/2 \leq t \leq t_i + \Delta_0$ and $\chi_t(\zeta)$ is an unstable segment of bounded length in $\varphi_t(\alpha)$, where (α, ℓ) is ltd with $[\varphi_t] \in \ell$. (4.14.1) also holds if $t = t_i + \Delta_0/2$, and $\chi_t(\zeta)$ is an unstable segment across $\varphi_t(\Omega \setminus \Omega')$, where Ω' is the union of all α such that (α, ℓ) is ltd for some $\ell \subset [y_{t_i+\Delta_0/2} y_{t_i+\Delta_0}]$, and Ω is the convex hull of Ω'. This is because, if σ is any component of $\Omega \setminus \Omega'$ — and hence σ is a disc or annulus – then $|\varphi_t(\partial\sigma)|$ is bounded. (4.14.1) is also true if $\chi_t(\zeta)$ is the union of a (possibly empty) segment across $\varphi_t(\Omega \setminus \Omega')$ and an adjacent segment $\chi_t(\zeta')$in $\varphi_t(\alpha)$ for an α such that (α, ℓ) is ltd with $\ell \subset [y_{t_i+\Delta_0/2}, y_{t_i+\Delta_0}]$, with $\chi_t(\zeta')|_t \geq |\chi_t(\zeta)_t/M$, and such that $\chi_t(\zeta')$ has Poincaré length at least multiple bounded form 0 injectivity radius at any point of $\chi_t(\zeta')$, for any $[\varphi_t] \in \ell$.

In what follows, we write $\chi_t(\zeta_i) \cap \varphi_t(\Omega)$ as

$$\cup_{\zeta \in A} \chi_t(\zeta)$$

for one, and hence any t, for a set A of such segments, that is, including a segment of some $\varphi_0(\alpha)$, where any two distinct segments in A have disjoint interiors.

The argument for finding (α_1, ℓ_1) is different from the argument for (α_i, ℓ_i) for $i > 1$. We take our fixed ltd parameters (Δ, r, s, K_0). Let ν_0 be as given by 3.4 and L be given by 3.5 for (Δ, r, s, K_0) and ν_0. Let L_0 be such that $L_j \leq L_0$ for all L_j arising as in 4.11. In what follows, we are going to apply 4.9 to 4.12 with $\Delta_1 = \Delta_0/3$. By 3.4, we can choose (α_1, ℓ_1) which is ltd with respect to (Δ, r, s, K_0), with $\ell_1 \subset [y_0, y_{\Delta_0}]$, such that $a'(\alpha_1) > c_0$, where c_0 depends only on the topological type of S. Now if $\alpha \cap \omega_j(\beta) \neq \emptyset$ for some β as in Property 5 and $u_j(\beta) \leq t_1 + \Delta_0/p \leq u_{j-1}(\beta)$ then $\alpha \cap \beta' \neq \emptyset$ for some $\beta' \supset \omega_j(\beta)$ with $a'(\beta') \leq e^{-\Delta_0/4}$ and $|\varphi_t(\partial\beta')| \leq e^{-\Delta_0/p}$ for $|t - t_1| \leq \Delta_0/p$. Then $\partial\beta' \cap \alpha \neq \emptyset$ if $\nu_0 > e^{-\Delta_)/p}$, as we can assume by taking Δ_0 sufficiently large, and $\alpha \subset \beta'$, which is impossible. On the other hand if $u_j(\beta) \geq t_1 + \Delta_0/p$ then we see that for any segment $\varphi_{t_1}(\gamma)$ across

$\varphi_{t_1}(\omega_j(\beta)\setminus\omega_{j+1}(\beta))$ we have $|\varphi_{t_1}(\gamma)|_{t_1}\leq e^{-\Delta_0/2p}|\zeta_1|$. Hence every segment $\varphi_{t_1}(\gamma)$ is adjacent to a segment $\zeta'\subset\zeta_1\setminus\varphi_{t_1}(\omega_j(\beta)\setminus\omega_{j+1}(\beta))$ on at least one side with $|\varphi_{t_1}(\gamma)\leq\mu_j|\zeta'|_{t_1}$, and hence Property 5 holds for ζ_1, provided that $e^{-\Delta_0/2p}<\mu_j/3$, which is true provided that Δ_0 is large enough given μ_j (for all j) and p.

Now given α_i and ζ_i, we need to find α_{i+1} and t_{i+1} and ζ_{i+1}. We use the inductively obtained properties of ζ_i, and the bounds this gives on segments $\zeta\subset\zeta_i$ for which $|\chi_t(\zeta)|$ is bounded for some $t>t_i$, as described above. We let B_j be the set of all segments γ of $\omega_j(\beta)\setminus\omega_{j+1}(\beta)$, for all β, such that $a'(\beta)\leq e^{-\Delta_0/3}$ and $u_0(\beta)\geq t_i$ and $|\varphi_t(\partial\beta)|<e^{-\Delta_0/p}$ for $|t-u_0(\beta)|\leq\Delta_0/p$. and $\varphi_t(\omega_j(\beta)\setminus\omega_{j+1}(\beta))\cap\chi_t(\zeta_i)\neq\emptyset$, (for one, hence any, t). Let B_j^1 be the set of all such segments γ such that $\varphi_{t_i}(\gamma)$ is not adjacent to a segment of $\chi_{t_i}(\zeta_i)$ which is outside $\varphi_{t_i}(\omega_j(\beta)\setminus\omega_{j+1}(\beta))$ and μ_i^{-1} times longer. Let

$$B_j^2=B_j\setminus B_j^1,\quad B^1=\cup_jB_j^1,\quad B^2=\cup_jB_j^2,\quad B=\cup_jB_j=B^1\cup B^2.$$

By the properties of the sets $\omega_j(\beta)\setminus\omega_{j+1}(\beta)$ described in 4.11, the number of elements of B_j^1 is bounded by the number of different (β,j), which, by 4.12, is N_j, where N_j depends only on $\mu_{j'}$ for $j'<j$. So, by choice of μ_j, we can take $N_j\mu_j^{1/2}$ as small as we like. By Property 5, we have, for any t,

$$\sum_{\gamma\in B^1}|\varphi_t(\gamma)\cap\chi_t(\zeta_i)|_t\leq\sum_j\mu_j^{1/2}N_j\cdot|\chi_t(\zeta_i)|_t.$$

By the properties of the set $\omega_j(\beta)\setminus\omega_{j+1}(\beta)$ of 4.11 we have

$$\sum_{\gamma\in B^2}|\varphi_t(\gamma)\cap\chi_t(\zeta_i)|_t\leq\sum_j\mu_jN_j\cdot|\chi_t(\zeta_i)|_t$$

We have

$$\chi_t(\zeta_i)\subset(\cup_{\gamma\in B}\varphi_t(B))\cup(\cup_{\zeta\in A}\chi_t(\zeta))$$

(for any t) and hence, since segments in A have disjoint interiors, we have

$$\sum_{\zeta\in A}|\chi_t(\zeta)|_t\geq|\chi_t(\zeta_i)|_t-\sum_{j\in B_j}|\varphi_t(\gamma)\geq\frac{1}{2}|\chi_t(\zeta_i)|_t.$$

assuming that μ_j is sufficiently small given N_j, for each j, that

$$\sum_jN_j\mu_j^{1/2}<\frac{1}{4}.$$

Now for A as above, let

$$A'=\{\zeta\in A,\gamma\in B^1:\chi_{t_i}(\zeta)\cap\varphi_{t_i}(\gamma)\neq\emptyset\}$$

then assuming that $e^{-\Delta_0/4}\leq\mu_j^{1/2}$ for all j and t,

$$\sum_{\zeta\in A'}|\chi_t(\zeta)|_t\leq2\sum_{\gamma\in B^1}|\varphi_t(\gamma)|_t$$

$$\leq 2\sum_j N_j\mu_j^{1/2}\cdot|\chi_t(\zeta_i)|_t.$$

For any $K>0$, if

$$A^{K,j}=\{\zeta\in A:|\chi_t(\zeta)\cap\varphi_t(\gamma)|_t\geq K|\chi_t(\zeta)|_t \text{ for some } \gamma\in B_j^2\}$$

then

$$\sum_{\zeta\in A^{K,j}}|\chi_t(\zeta)|_t\leq K^{-1}\sum_{\zeta\in A^{K,j}}|\chi_t(\zeta)\cap(\cup_{\gamma\in B_j^2}\varphi_t(\gamma))|_t$$

$$\leq K^{-1}\sum_{\gamma\in B_j^2}\varphi_t(\gamma))|_t\leq K^{-1}N_j\mu_j|\chi_t(\zeta)|_t.$$

Now put $A^j=A^{K,j}$ for $K=\mu_j^{1/2}/M$, for M as in 4.13. We have

$$\sum_{\zeta\in A^j}|\chi_t(\zeta)|_t\leq N_jM\mu_j^{1/2}|\chi_t(\zeta_i)|_t.$$

So

$$\sum_j\sum_{\zeta\in A^j}|\chi_t(\zeta)|_t+\sum_{\zeta\in A'}|\chi_t(\zeta)|_t\leq(2+M)\sum_j N_j\mu_j^{1/2}\cdot|\chi_t(\zeta_i)|_t$$

$$\leq 2(2+M)\sum_j N_j\mu_j^{1/2}\cdot\sum_{\zeta\in A}|\chi_t(\zeta)|_t.$$

So assuming that μ_j is big enough given N_j for each j, and given M, so that

$$4(2+M)\sum_j N_j\mu_j^{1/2}<1,$$

we have

$$A\setminus(A'\cup_j A^j)\neq\emptyset.$$

We choose any $\zeta\in A\setminus(A'\cup_j A^j)$ and, using the definition of A, let (α,ℓ) be any ltd with $\ell\subset[y_{t_i+\Delta_0/2},y_{t_i+\Delta_0}]$ and let ζ' be such that $\chi_t(\zeta')\subset\chi_t(\zeta)\cap\varphi_t(\alpha)$ for any t and $\chi_t(\zeta')$ has Poincaré length which is boundedly proportional to the injectivity radius for some $y_t=[\varphi_t]\in\ell$. By the definition of segments of A, such an (α,ℓ) does exist. Then, since $\zeta\notin A'$,

$$\chi_t(\zeta')\cap\varphi_t(\gamma)=\emptyset \text{ for all } \gamma\in B^1,$$

and since $\zeta\notin A^j$ for any j, we have, for all $\gamma\in B_j^2$, for any j, and any t

$$|\chi_t(\zeta')\cap\varphi_t(\gamma)|_t\leq\sqrt{\mu_j}|\chi_t(\zeta')|_t.$$

Then we take $\alpha=\alpha_{i+1}$, $\ell=\ell_{i+1}$ and $\zeta_{i+1}=\zeta'$, and choose $t_{i+1}=t$ so that $y_t\in\ell$ and $\chi_t(\zeta')$ has Poincaré length which is boundedly proportional to the injectivity radius. Then all the required properties hold for $(\alpha_{i+1},\ell_{i+1})$, t_{i+1} and ζ_{i+1}, including Property 5. □

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